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One Time Periodic Solutions of Navier-Stokes Equations using Asymptotic Stability and Bifurcations

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Abstract:
This article demonstrated an asymptotic dependability basis for the arrangements of Primitive conditions defined on a three-dimensional finite barrel-shaped space with time-subordinate compelling terms. Under a reasonable littleness presumption on the nontrivial driving terms, we get the presence of the time occasional answer for the Primitive conditions. Also, this time-occasional arrangement is asymptotically steady in L2 sense.
Keywords: Asymptotoc stability, Bifurcations, Navier-Stokes Equations

Introduction

The Primitive conditions of the huge scope sea are gotten from the Navier-Stokes conditions from Coriolis power, combined with thermal dynamic condition and saltiness diffusing-transport condition under the Boussinesq and hydrostatic approximations. From numerical perspectives, individuals were persuaded that the issue of worldwide well-posedness of solid answers for 3D gooey Primitive conditions may be as hard as the 3D Navier-Stokes conditions or considerably more difficult. For sure, by considering the hydrostatic estimation, the vertical speed in the Primitive conditions turns into a demonstrative variable due to the divergence free condition. Along these lines, the nonlinear term for the incompressible Navier-Stokes conditions has the form: velocity first-request subsidiaries of speed while the Primitive conditions have a more convoluted structure: first-request subordinates of level speed first-request subsidiaries of even speed.

The central issue is that the obscure weight (the surface weight) is basically a component of two-dimensional even factors. The creators exploit the central issue to build up the pivotal L6 gauge for the speed in the confirmation of the worldwide well-posedness of solid arrangements. For quite a few years, asymptotic strength issues for fluid movements under different sorts of

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settings have pulled in a ton of consideration. Roused by a progression of papers of Serrin, numerous creators have given to the investigation of the presence of time-occasional arrangements of Navier-Stokes conditions under different settings. We allude the intrigued perusers to on account of limited spaces and on account of unbounded areas. In, under the presumption of the presence of worldwide in time arrangements of 3d Navier-Stokes conditions, Serrin gave a widespread measure of the asymptotic soundness of the speed fields. In particular, assume the fluid with the consistency ν and maximal speed V is confined in a limited three-dimensional space with breadth d. At that point, the flow with the Reynolds number $V d/\nu$ under is asymptotically steady in L2 sense. In a proceeded with work, Serrin demonstrated the presence of time-intermittent arrangements with period T under the further suspicions:

(1) The driving term is time-occasional with period T, and

(2) There exists a flow with Reynolds number under and this flow is equicontinuous in space variable forever.

The asymptotic solidness of e conditions defined on a finite tube-shaped space with time subordinate compelling terms. At the point when appropriate littleness conditions are forced on the driving terms, we demonstrate that the L2 standard of the difference of any two in number arrangements will remain in general zero dramatically. As a rule, it is difficult to infer a rot gauge of the L2(M) standard of the difference of two discretionary flows. Be that as it may, the conditions' convection structure permits us to infer such a gauge between a subjective flow and a flow with certain diminutiveness. In this manner, we can utilize the triangle disparity to get the rot gauge of the difference of two subjective flows. At that point, we demonstrate the presence and time-intermittent arrangement under the supposition that the driving capacities are time-occasional and little. The littleness in the above explanations relies upon the ν_1,ν_2,μ_1,μ_2, the limit condition α and the size of the area.

As a fundamental work for investigating the asymptotic soundness of Primitive conditions with time-subordinate driving, we follow the thoughts of and demonstrate the worldwide in time presence of answers for Primitive conditions in our setting. We next exploit the uniform lemma to get the appraisals to build up a coupled framework of conventional differential imbalances referring the energy appraisals of fluid speed and the temperature work. Under a reasonable diminutiveness condition on the constraining term, the asymptotic solidness of arrangements with little beginning information is then gotten from this arrangement of differential imbalances. Under a comparative setting, in Tachim treated the presence of time-intermittent arrangements of the Primitive conditions by Galerkin's technique under a generally more grounded supposition that the warmth source is differentiable in transient variable and fulfilling some diminutiveness conditions. Notwithstanding, the creator didn't address the security issue in. In this article, we loosen up the routineness prerequisite of driving terms and we give an

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asymptotic strength investigation. The thought we utilize to demonstrate the presence and uniqueness of time occasional solid arrangements depends on a Serrin's strategy, which we think all the more intelligently direct and numerically excellent. It is worth-referenced that our investigation can be applied to two-dimensional Navier-Stokes conditions combined with heat diffusion conditions on limited areas. It should be simple for intrigued perusers to supply the essential subtleties.
The Primitive conditions and their varieties are that we restate imbalances, uniform disparity and some significant numerical outcomes on Primitive conditions. The worldwide in time presence of the arrangement and a vital lemma are expressed and demonstrated. At long last, we state and demonstrate our primary hypotheses. Investigation for different spaces, such as the round shell area or unbounded area, should be tended to elsewhere.
MODEL EQUATION:

$$\partial v\ \partial t + (v\cdot\nabla)v + w\ \partial v\ \partial z + f v \perp + \nabla p = \nu 14 v + \mu 1 \partial 2 v + \partial z 2 + F1$$

$$\nabla p = \nu 14 v + \mu 1 \partial 2 v + \partial z 2 + F1 + w\ \partial\theta\ \partial z = \nu 24\theta + \mu 2\ \partial 2\theta\ \partial z 2 + F2$$

$$\partial v\ \partial z = 0$$
$$w = 0$$
$$\partial\theta\,\partial z = -\alpha\theta$$

on Γb :

$$\partial v\,\partial z = 0,$$
$$\partial\theta\,\partial z = 0$$

on Γl :

$$v\cdot\sim n = 0,$$
$$\partial v\,\partial\sim n \times \sim n = 0, \theta = 0.$$

Here, α = 0 is a given consistent and ~n is the inward unit typical vector to Γl. We comment that no wind-driven limit conditions are forced on the lower surface; free-slip and no warmth flux limit conditions are forced on the horizontal limit and base, as well. The underlying condition viable is given by:

$$(v,\theta)(0) = (v0, \theta 0).$$
$$w(x,y,z,t) = -Z\ z - h\nabla\cdot v(x,y,\xi,t)d\xi + \partial v\ \partial t + (v\cdot\nabla)v$$
$$p(x,y,z,t) = p0(x,y,t) - Z\ z - h\theta(x,y,\xi,t)d\xi$$
$$\partial v\ \partial t + (v\cdot\nabla)v - (Z\ z - h\nabla\cdot v(x,y,\xi,t)d\xi)\partial v\ \partial z + f v \perp$$
$$+\nabla p0 - \nabla(Zz - h\theta(x,y,\xi,t)d\xi)$$
$$= \nu 14 v + \mu 1\partial 2 v\ \partial z 2 + F1$$
$$\partial\theta\partial t + (v\cdot\nabla)\theta - (Z\ z - h\nabla\cdot v(x,y,\xi,t)d\xi)\partial\theta\ \partial z = \nu 24\theta + \mu\partial 2\theta\ \partial z 2 + F2,$$
$$\partial v\ \partial z = 0$$
$$v = 1\ hZ\ 0 - hv(x,y,z,t)dz, \tilde{}\ v = v - \bar{}\ v.$$

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We comment that the variation ¯ v is meant to the barotropic mode and the variable ˜ v is signified to the barochostic mode. As have been seen in we see that ¯ v and ˜ v fulfill the accompanying conditions:

$$\partial^{-} v \partial t + (^{-} v \cdot \nabla)^{-} v + (^{\sim} v \cdot \nabla)^{\sim} v + (\nabla \cdot {}^{\sim} v)^{\sim} v + f^{-} v \perp \quad +\nabla p0 - Z z - h \nabla\theta(x, y, \xi, t) d\xi = v^{-} v + {}^{-} F1,$$
$$v \cdot \sim n| \partial M0 = 0,$$
$$\partial^{-} v \partial \sim n \times \sim n|\partial M0 = 0, \text{ and}$$
$$\partial^{\sim} v \partial t + (^{\sim} v \cdot \nabla)^{\sim} v - (Z z - h\nabla \cdot {}^{\sim} v(x, y, \xi, t) d\xi)\partial^{\sim} v \partial z v^{-} v + {}^{-} F1,$$
$$+(^{\sim} v \cdot \nabla)^{-} v + (^{-} v \cdot \nabla)^{\sim} v - (^{\sim} v \cdot \nabla)^{\sim} v + (\nabla \cdot {}^{\sim} v)^{\sim} v \; f^{\sim} v \perp$$
$$-h\nabla\theta(x, y, \xi, t) d\xi + Z z - h \nabla\theta(x, y, \xi, t) d\xi \; v^{-} v + {}^{-} F1,$$
$$= v14^{\sim} + \mu 1 \; \partial^{\sim} v \partial z + {}^{\sim} F1, \partial^{\sim} v \partial z$$

Notice that there are three differences between our presumptions from those introduced in. To start with, rather than a Neumann limit condition, we force a Dirichlet limit state of temperature work θ on the horizontal limit Γu. The explanation is that we need a Poincare imbalance of θ on Γu for L4(M) gauge of ∂θ/∂z. Also, while we let F be time subordinate, the creators of, let F1 = 0 and F2 be time free. Third, for the underlying information, other than the condition, we add the suppositions and. The nearby presence and uniqueness of the solid answer for F ∈L ∞(0,T;L2(M)) have been demonstrated:

$$|\theta 3 \, z|L4 = |\theta 2 \, z|3/2 \; L6 \le (|\theta 2 \, z|L2 + |\nabla 3\theta 2 \, z|L2) \le$$
$$|\theta z|3 \; L4 + \_x0010\_ZM\theta 2 \, z|\nabla\theta z|2dM + ZM\theta 2 \, z\theta 2 \, zzdM3|\theta 3 \, z|L2 = |\theta 2 \, z|3/2 \; L3 \le$$
$$c\_x0010\_|\theta z|3 \; L4 + ZM\theta 2 \, z|\nabla\theta z|2dM + ZM\theta 2 \, z| \; \partial\theta z \; \partial z \; |2dM \; L2(\Gamma u) =$$
$$|\theta 2|3/2 \; L3(\Gamma u) \le c\_x0010\_|\theta 2|2/3 \; L3(\Gamma u)|\theta 2|1/3 \; H1(\Gamma u)3/2 \le$$
$$\_x0010\_|\theta|4/3 \; L4(\Gamma u)(|\theta|2/3 \; L4(\Gamma u) + |\theta|\nabla\theta||1/3 \; L2(\Gamma u))3/2 \le L4(\Gamma u) +$$
$$|\theta|2 \; L4(\Gamma u)|\theta|\nabla\theta||1/2 \; L2(\Gamma u).$$

$$|ZM(vz \cdot \nabla)\theta\theta 3 \, zdM| \le |vz|L4|\nabla\theta|L2|\theta 3 \, z|L4 \le |vz|L4|\nabla\theta|Lz|3 \; L4 +$$
$$ZM\theta 2 \, z|\nabla\theta z|2dM + ZM\theta 2 \, z\theta 2 \, zzdM(\nabla \cdot v)\theta 4 \, zdM| \le |\nabla v|L2|\theta z|L4|\theta 3 \, z|L4 \le$$
$$|\nabla v|L2|\theta z|L4\_x0010\_|\theta z|3 \; L4 + (ZM\theta 2 \, z|\nabla\theta z|2dM +$$
$$ZM\theta 2 \, z\theta 2, |ZM \; \partial F2 \; \partial z \; \theta 3 \, zdM| \le |\partial F2 \; \partial z|L2|\theta 3 \, z|L2 \le$$
$$c| \; \partial F2 \; \partial z \; |L2\_x0010\_|\theta z|3 + ZM\theta 2 \, z|\nabla\theta z|2dM +$$
$$ZM\theta 2 \, z\theta 2 \, zzdM3/4, |Z\Gamma u \; F2\theta 3dM0| \le$$
$$c|F2|L2(\Gamma u)\theta|3 \; L4(\Gamma u) + |\theta|2 \; L4(\Gamma u)|\theta|\nabla\theta||1/2 \; L2(M), \le$$
$$c|F2|L2(\Gamma u)|\theta|4 \; L4(\Gamma u) + v2|\theta|\nabla|\theta||2 \; L2(\Gamma u) + c|F2|2 \; L2(\Gamma u) \quad .$$

As for the estimate of ∇v in L2(M), by taking L2(M) inner product of with −4v and noting that

$$|ZM(v \cdot \nabla)v \cdot 4v \, dM| \le |v|L6|4v|L|\nabla v|L3 \le c|v|L6|4v|L|\nabla(|4v|1/2 \; L +$$
$$|\nabla \; \partial v \; \partial z|1/2 \; L) \le v1 \; 10|4v|2 \; L2 + \mu 1 \; 2 \; |\nabla \; \partial v \; \partial z|2 \; L + c|v|4 \; L6|\nabla v|2 \; L,$$

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$|ZM\_x001\_Z\ z - h\ \nabla \cdot v\ d\xi \partial v\ \partial z\ \cdot 4v\ dM| \leq cZM0\_x001\_Z0 - h\ |\nabla v| d\xi Z\ 0\ - h\ |\ \partial v\ \partial z||4v| d\xi dxdy \leq c|\nabla v|1/2\ L|\ \partial v\ \partial z|1/2\ L\ |\nabla\ \partial v\ \partial z|1/2\ L2\ |4v|3/2\ L2 \leq v1\ 10|4v|2\ L2\ +\ c|\nabla v|2\ L2|\nabla \partial v\ \partial z|2\ L2|\partial v\ \partial z|2\ L2,$

$|ZM\nabla\_x0012\_Z\ z\ - h\theta d\xi 4v\ dM| \leq\ ZM(Z\ 0\ - h$
$|\nabla \theta| d\xi)|4v| dM \leq\ c|\nabla \theta|L2|4v|L2 \leq v1\ 10|4v|2\ L2 + c|\nabla \theta|2\ L, |ZMfv \perp$
$4vdM| \leq c|v|2\ L2 + 1\ 10|4v|2\ L2, |ZMF1$
$(-4v)dM| \leq c|F1|L2|4v|L2\ \leq\ v1\ 10|4v|2\ L2 + c\ v1|x0012\_Z\ z - h\theta d\xi 4v\ dM|$
$\leq\ ZM(Z\ 0\ - h)$

we reach

$$\begin{aligned} &d\ dt|\nabla v|2\ L2 + v1|4v|2\ L2 + \mu1|\nabla \partial v\ \partial z|2\ L2 \\ &\leq c(1\ + |v|4\ L6\ + |\partial v\ \partial z|2L2|\nabla \partial v\ \partial z|2\ L2)|\nabla v|2\ L2\ + c|\nabla \theta|2\ L2 \\ &+ c|F1|2\ L\infty(0,T;(L2(M))|\nabla v|2\ L2\ + Zt0\_x0012\_v1|4v|2\ L2 \\ &+ \mu1|\nabla \partial v\ \partial z|2\ L2s \leq J6(t), \end{aligned}$$

Where

$$J6(t) =\ ec(1 + J2\ 4 + J2/3\ 3 + J2\ 5)t\_x0010\_|v0|2\ H1(M)2 + cJ1(t)t$$

$$\begin{aligned} &|ZMv \cdot \nabla \theta(4\theta\ +\ \partial 2\theta\ \partial z2)dM|\ \leq\ cZM|v||\nabla \theta|(|4\theta| + |\partial 2\theta\ \partial z2|)dM \\ &\leq c|v|L6(|4\theta|L2\ + |\ \partial 2\theta\ \partial z2|L2)|\nabla \theta|L3 \\ &\leq\ c|v|L6(|4\theta|L2\ + |\ \partial 2\theta\ \partial z2|L2)|\nabla \theta|1/2\ L2\ (|4\theta \nabla \partial v\ \partial z|1/2\ L2\ ) \\ &\leq v2\ 6|4\theta|2\ L2 + v2 + \mu2\ 4\ |\nabla \partial v\ \partial z|2\ L2\ + \mu2\ 6|\partial 2\theta\ \partial z2|L2 \\ &+ c|v|4\ L6|\nabla \theta|2\ L2, \\ &|ZM\_x0012\_Z\ z - h\nabla \cdot v\ d\xi \partial \theta\ \partial z(4\theta + \partial 2\theta\ \partial z2)dM| \\ &\leq\ ZM0\_x0012\_Z\ 0 - h|\nabla v| d\xi Z\ 0\ - h|\ \partial \theta\ \partial z|(|4\theta| \\ &+ |\partial 2\theta\ \partial z2|)d\xi dM0 \\ &\leq\ c|\nabla v4v|1/2\ L2\ |\partial \theta\ \partial z|1/2L2|\nabla\ \partial \theta\ \partial z|1/2\ L2\ (|4\theta|L2 \\ &+ |\partial 2\theta\ \partial z2|L2) \\ &\leq v2\ 6\ |4\theta|2\ L2 + \mu2\ 6\ |\partial 2\theta\ \partial z2|2L2 + \mu2 \\ &+ v2\ 4\ |\nabla \partial \theta\ \partial z|2\ L2\ c|\nabla v|2\ L2|4v|2\ L2|\partial \theta\ \partial z|2\ L2, \end{aligned}$$

$$\begin{aligned} &|ZMF2(4\theta + \partial 2\theta\ \partial z2)dM| \leq\ c|F2|L2(|4\theta|L2 + |\partial 2\theta\ \partial z2|L2) \\ &\leq v2\ 6\ |4\theta|L2 + \mu2\ 6\ |\partial 2\theta\ \partial z2|L2\ + c|F2|2\ L\infty(0,T;L2(M)), \end{aligned}$$

we obtain

$$\begin{aligned} &+d\ dt\_x0012\_|\nabla \theta|2\ L2 + |\partial \theta\ \partial z|2\ L2 + \alpha|\nabla \theta|2\ L2(\Gamma u) + v1|4\theta|2\ L2\ + (v1 \\ &+ \mu1)\_x0012\_|\nabla \partial \theta\ \partial z|2\ L2\ +\ \alpha|\nabla \theta|2\ L2(\Gamma u) + \mu1|\partial 2\theta\ \partial z2|2\ L2 \\ &\leq c|v|4\ L6\ + |\nabla v|2\ L2|4v|2\ L2\_x0012\_|\nabla \theta|2\ L2\ + |\ \partial \theta\ \partial z|2\ L2 \\ &+ c|F2|2\ L\infty(0,T;L2(M)). \end{aligned}$$

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|θ3 z|L4 =
|θ2 z|3/2 L6 ≤ (|θ2 z|L2 + |∇3θ2 z|L2) ≤
|θz|3 L4 + _x0010_ZMθ2 z|∇θz|2dM + ZMθ2 zθ2 zzdM3|θ3 z|L2 = |θ2 z|3/
2 L3 ≤
c_x0010_|θz|3 L4 + ZMθ2 z|∇θz|2dM + ZMθ2 z| ∂θz ∂z |2dM L2(Γu) =
|θ2|3/2 L3(Γu) ≤ c_x0010_|θ2|2/3 L3(Γu)|θ2|1/3 H1(Γu)3/2 ≤
_x0010_|θ|4/3 L4(Γu)(|θ|2/3 L4(Γu) + |θ|∇θ||1/3 L2(Γu))3/2 ≤ L4(Γu) +
|θ|2 L4(Γu)|θ|∇θ||1/2 L2(Γu).
|ZM(vz · ∇)θθ3 zdM| ≤ |vz|L4|∇θ|L2|θ3 z|L4 ≤ |vz|L4|∇θ|Lz|3 L4 +
ZMθ2 z|∇θz|2dM + ZMθ2 zθ2 zzdM(∇ · v)θ4 zdM| ≤ |∇v|L2|θz|L4|θ3 z|L4 ≤
|∇v|L2|θz|L4_x0010_|θz|3 L4 + (ZMθ2 z|∇θz|2dM +
ZMθ2 zθ2 , |ZM ∂F2 ∂z θ3 zdM| ≤ | ∂F2∂z |L2|θ3 z|L2 ≤
c| ∂F2 ∂z |L2_x0010_|θz|3 + ZMθ2 z|∇θz|2dM +
ZMθ2zθ2 zzdM3/4, |ZΓu F2θ3dM0| ≤
c|F2|L2(Γu)θ|3 L4(Γu) + |θ|2 L4(Γu)|θ|∇θ||1/2 L2(M), ≤
c|F2|L2(Γu)|θ|4 L4(Γu) + ν2|θ|∇|θ||2 L2(Γu) + c|F2|2 L2(Γu) +
c|F2|L2(Γu).

Gronwall inequality, we obtain

|∇θ|2 L2 + | ∂θ ∂z|2 L2 + α|∇θ|2 L2(Γu) + ν1Z t 0 |4θ|2 L2ds + Z t 0 (ν1 +
µ1)_x0012_|∇∂θ ∂z|2 L2 + α|∇θ|2 L2(Γu) + µ1|∂2θ ∂z2|2 L2ds ≤
J7(t), (3.51) where J7(t) =
ec(1 + J2 4 + J2/3 3)t + J2 6(t)_x0010_|θ0|2 H1(M) +
ct|F2|2 L∞(0, T; L2(M)).

|θ3 z|L4 = |θ2 z|3/2 L6 ≤ (|θ2 z|L2 + |∇3θ2 z|L2) ≤
|θz|3 L4 + _x0010_ZMθ2 z|∇θz|2dM + ZMθ2 zθ2 zzdM3|θ3 z|L2 = |θ2 z|3/
2 L3 ≤
c_x0010_|θz|3 L4 + ZMθ2 z|∇θz|2dM + ZMθ2 z| ∂θz ∂z |2dM L2(Γu) =
|θ2|3/2 L3(Γu) ≤ c_x0010_|θ2|2/3 L3(Γu)|θ2|1/3 H1(Γu)3/2 ≤
_x0010_|θ|4/3 L4(Γu)(|θ|2/3 L4(Γu) + |θ|∇θ||1/3 L2(Γu))3/2 ≤ L4(Γu) +
|θ|2 L4(Γu)|θ|∇θ||1/2 L2(Γu).
|ZM(vz · ∇)θθ3 zdM| ≤ |vz|L4|∇θ|L2|θ3 z|L4 ≤ |vz|L4|∇θ|Lz|3 L4 +
ZMθ2 z|∇θz|2dM + ZMθ2 zθ2 zzdM(∇ · v)θ4 zdM| ≤ |∇v|L2|θz|L4|θ3 z|L4 ≤
|∇v|L2|θz|L4_x0010_|θz|3 L4 + (ZMθ2 z|∇θz|2dM +
ZMθ2 zθ2 , |ZM ∂F2 ∂z θ3 zdM| ≤ | ∂F2 ∂z |L2|θ3 z|L2 ≤
c| ∂F2 ∂z |L2_x0010_|θz|3 + ZMθ2 z|∇θz|2dM +
ZMθ2 zθ2 zzdM3/4, |ZΓu F2θ3dM0| ≤
c|F2|L2(Γu)θ|3 L4(Γu) + |θ|2 L4(Γu)|θ|∇θ||1/2 L2(M), ≤
c|F2|L2(Γu)|θ|4 L4(Γu) + ν2|θ|∇|θ||2 L2(Γu) + c|F2|2 L2(Γu) +
c|F2|L2(Γu).

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< (𝑣 ⋅ 𝛻)𝑢 − 𝑍 𝑧 − ℎ𝛻 ⋅ 𝑣 𝑑𝜉𝜕𝑢 𝜕𝑧|𝑢|2𝑢 >= 0,< 𝜈14𝑢 + μ1𝜕2𝑢 𝜕2𝑧
|𝑢|2𝑢 >=
−𝑍𝑀𝜈1|𝑢|2|𝛻𝑢|2 + μ1|𝑢|2|𝜕𝑢 𝜕𝑧| + 𝜈1 2 |𝛻|𝑢|2|2 + μ12|𝜕 𝜕𝑧|𝑢|2|2𝑑𝑀,
< 𝑓𝑢 ⊥, |𝑢|2𝑢 >= 0, |𝑍𝑀[(𝑢 ⋅ 𝛻)𝑣 − (𝛻 ⋅ 𝑣)𝑢] ⋅ |𝑢|2𝑢𝑑𝑀| ≤
𝑐𝑍𝑀|𝑣||𝛻𝑢||𝑢|3 𝑑𝑀 ≤ 𝑐||𝛻𝑢||𝑢||𝐿2||𝑢|2|𝐿3|𝑣|𝐿6 ≤ 𝑐||𝛻𝑢||𝑢||𝐿2||𝑢|2|1/
2 𝐿2_𝑥0012_|𝛻|𝑢|2|1/2 𝐿2 + |𝜕 𝜕𝑧|𝑢|2|1/2 𝐿2 + ||𝑢|2|1/2 𝐿2|𝑣|𝐿6
= 𝑐||𝛻𝑢||𝑢||𝐿2|𝑢|𝐿4_𝑥0012_|𝛻|𝑢|2|1/2 𝐿2 + |𝜕 𝜕𝑧|𝑢|2|1/2 𝐿2 + |𝑢|𝐿4|𝑣|𝐿6 ≤
𝑐(|𝑣|2 𝐿6 + |𝑣|4 𝐿6)|𝑢|4 𝐿4 + 𝜈1 4 ||𝛻𝑢||𝑢||2 𝐿2 + 𝜈18 |𝛻|𝑢|2|2 𝐿2 +
μ1 8 |𝜕 𝜕𝑧|𝑢|2|2 𝐿2, | < |𝜃3 𝑧|𝐿4 = |𝜃2 𝑧|3/2 𝐿6 ≤ (|𝜃2 𝑧|𝐿2 + |𝛻3𝜃2 𝑧|𝐿2) ≤
|𝜃𝑧|3 𝐿4 + _𝑥0010_𝑍𝑀𝜃2 𝑧|𝛻𝜃𝑧|2𝑑𝑀 + 𝑍𝑀𝜃2 𝑧𝜃2 𝑧𝑧𝑑𝑀3|𝜃3 𝑧|𝐿2 = |𝜃2 𝑧|3/
2 𝐿3 ≤ 𝑐_𝑥0010_|𝜃𝑧|3 𝐿4 + 𝑍𝑀𝜃2

𝑧|𝛻𝜃𝑧|2𝑑𝑀 + 𝑍𝑀𝜃2 𝑧| 𝜕𝜃𝑧 𝜕𝑧 |2𝑑𝑀 𝐿2(𝛤𝑢) = |𝜃2|3/2 𝐿3(𝛤𝑢) ≤
𝑐_𝑥0010_|𝜃2|2/3 𝐿3(𝛤𝑢)|𝜃2|1/3 𝐻1(𝛤𝑢)3/2 ≤ _𝑥0010_|𝜃|4/3 𝐿4(𝛤𝑢)(|𝜃|2/
3 𝐿4(𝛤𝑢) + |𝜃|𝛻𝜃||1/3 𝐿2(𝛤𝑢))3/2 ≤ 𝐿4(𝛤𝑢) + |𝜃|2 𝐿4(𝛤𝑢)|𝜃|𝛻𝜃||1/2 𝐿2(𝛤𝑢).

|𝑍𝑀(𝑣𝑧 ⋅ 𝛻)𝜃𝜃3 𝑧𝑑𝑀| ≤ |𝑣𝑧|𝐿4|𝛻𝜃|𝐿2|𝜃3 𝑧|𝐿4 ≤ |𝑣𝑧|𝐿4|𝛻𝜃|𝐿𝑧|3 𝐿4 +
𝑍𝑀𝜃2 𝑧|𝛻𝜃𝑧|2𝑑𝑀 + 𝑍𝑀𝜃2 𝑧𝜃2 𝑧𝑧𝑑𝑀(𝛻 ⋅ 𝑣)𝜃4 𝑧𝑑𝑀| ≤ |𝛻𝑣|𝐿2|𝜃𝑧|𝐿4|𝜃3 𝑧|𝐿4 ≤
|𝛻𝑣|𝐿2|𝜃𝑧|𝐿4_𝑥0010_|𝜃𝑧|3 𝐿4 + (𝑍𝑀𝜃2 𝑧|𝛻𝜃𝑧|2𝑑𝑀 +
𝑍𝑀𝜃2 𝑧𝜃2 , |𝑍𝑀 𝜕𝐹2 𝜕𝑧 𝜃3 𝑧𝑑𝑀| ≤ | 𝜕𝐹2 𝜕𝑧 |𝐿2|𝜃3 𝑧|𝐿2 ≤
𝑐| 𝜕𝐹2 𝜕𝑧 |𝐿2_𝑥0010_|𝜃𝑧|3 + 𝑍𝑀𝜃2 𝑧|𝛻𝜃𝑧|2𝑑𝑀 +
𝑍𝑀𝜃2 𝑧𝜃2 𝑧𝑧𝑑𝑀3/4, |𝑍𝛤𝑢 𝐹2𝜃3𝑑𝑀0| ≤
𝑐|𝐹2|𝐿2(𝛤𝑢)𝜃|3 𝐿4(𝛤𝑢) + |𝜃|2 𝐿4(𝛤𝑢)|𝜃|𝛻𝜃||1/2 𝐿2(𝑀), ≤
𝑐|𝐹2|𝐿2(𝛤𝑢)|𝜃|4 𝐿4(𝛤𝑢) + 𝜈2|𝜃|𝛻|𝜃||2 𝐿2(𝛤𝑢) + 𝑐|𝐹2|2 𝐿2(𝛤𝑢) +
𝑐|𝐹2|𝐿2(𝛤𝑢) 𝜕𝐹1 𝜕𝑧

|𝑢|2𝑢 > | ≤ |𝜕𝐹1 𝜕𝑧 |𝐿2||𝑢|3|𝐿2 = |𝜕𝐹1 𝜕𝑧 |𝐿2||𝑢|2|3/2 𝐿3 ≤
𝑐|𝜕𝐹1𝜕𝑧|𝐿2_𝑥0012_||𝑢|2|1/2 𝐿2 (|𝛻|𝑢|2|1/2 𝐿2 + |𝜕 𝜕𝑧|𝑢|2|1/2 𝐿2)3/2 ≤
𝑐|𝜕𝐹1 𝜕𝑧 |𝐿2|𝑢|3/2 𝐿4 (|𝛻|𝑢|2|3/4 𝐿2 + | 𝜕 𝜕𝑧|𝑢|2|3/4 𝐿2) ≤
𝜈1 8 |𝛻|𝑢|2|2 𝐿2 + μ1 8 |𝜕 𝜕𝑧|𝑢|2|2 𝐿2 + 𝑐|𝜕𝐹1 𝜕𝑧|8/5 𝐿2 |𝑢|12/5 𝐿4 ≤
𝜈1 8 |𝛻|𝑢|2|2 𝐿2 + μ1 8 |𝜕 𝜕𝑧|𝑢|2|2 𝐿2 + 𝑐|𝜕𝐹1 𝜕𝑧 |𝐿∞(0, 𝑇; 𝐿2(𝑀)2) +
|𝜕𝐹1 𝜕𝑧 |2 𝐿∞(0, 𝑇; 𝐿2(𝑀)2)|𝑢|4 𝐿4,
|𝑍𝑀𝛻𝜃 ⋅ |𝑢|2𝑢𝑑𝑀| ≤ 𝑐𝑍𝑀|𝜃||𝛻𝑢||𝑢|2 𝑑𝑀 ≤ 𝑐|𝜃|𝐿4|𝑢|𝐿4||𝑢||𝛻𝑢||𝐿2 ≤
𝜈1 4 ||𝑢||𝛻𝑢||2 𝐿2 + 𝑐|𝜃|2 𝐿4 + |𝜃|2 𝐿4|𝑢|4 𝐿4.

L4 Estimate for ∂v/∂z To fill the second gap, we need to perform L4(M) estimate of ∂v/∂z. For that purpose, we take L2(M) inner product of (3.44) with |u|2u and use the following facts:
< (𝑣 ⋅ 𝛻)𝑢 − 𝑍 𝑧 − ℎ 𝛻 ⋅ 𝑣 𝑑𝜉 𝜕𝑢 𝜕𝑧 , |𝑢|2𝑢 >= 0, <
𝜈14𝑢 + μ1 𝜕2𝑢 𝜕2𝑧 , |𝑢|2𝑢 >=
−𝑍𝑀𝜈1|𝑢|2|𝛻𝑢|2 + μ1|𝑢|2|𝜕𝑢 𝜕𝑧| + 𝜈1 2 |𝛻|𝑢|2|2 + μ1 2 | 𝜕 𝜕𝑧|𝑢|2|2𝑑𝑀, <
𝑓𝑢 ⊥, |𝑢|2𝑢 >= 0, |𝑍𝑀[(𝑢 ⋅ 𝛻)𝑣 − (𝛻 ⋅ 𝑣)𝑢] ⋅ |𝑢|2𝑢𝑑𝑀| ≤

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$cZM|v||\nabla u||u|3\, dM \ \le\ c||\nabla u||u||L2||u|2|L3|v|L6 \ \le\ c||\nabla u||u||L2||u|2|1/$
$2\, L2$ _x0012_ $|\nabla|u|2|1/2\, L2\ +|\,\partial\, \partial z|u|2|1/2\, L2\ +||u|2|1/2\, L2|v|L6\ =$
$c||\nabla u||u||L2|u|L4$_x0012_$|\nabla|u|2|1/2\, L2 + |\,\partial\,\partial z|u|2|1/2\, L2 + |u|L4|v|L6\ \le$
$c(|v|2\, L6 + |v|4\, L6)|u|4\, L4 + \nu 1\ 4\ ||\nabla u||u||2\, L2 + \nu 1\ 8\ |\nabla|u|2|2\, L2\ +$
$\mu 1\ 8\,|\,\partial\,\partial z|u|2|2\, L2, |<\ \partial F1\, \partial z\, , |u|2u\ >\ | \le |\,\partial F1\, \partial z\,|L2||u|3|L2\ =$
$|\,\partial F1\, \partial z\,|L2||u|2|3/2\, L3\ \le\ c|\,\partial F1\, \partial z\,|L2$_x0012_$||u|2|1/2\, L2\ (|\nabla|u|2|1/2\, L2 +$
$|\,\partial\, \partial z|u|2|1/2\, L2\,)3/2 \le c|\,\partial F1\, \partial z\,|L2|u|3/2\, L4\ (|\nabla|u|2|3/4\, L2 + |\,\partial\, \partial z|u|2|3/$
$4\, L2\,) \le \nu 1\ 8\,|\nabla|u|2|2\, L2\ +\ \mu 1\ 8\,|\,\partial\,\partial z|u|2|2\, L2 + c|\,\partial F1\, \partial z\,|8/5\, L2\,|u|12/$
$5\, L4 \le \nu 1\ 8\,|\nabla|u|2|2\, L2 + \mu 1\ 8|\partial\, \partial z|u|2|2\, L2\ +\ c|\,\partial F1\, \partial z\,|L\infty(0,T;L2(M)2) +$
$|\,\partial F1 \partial z|2\, L\infty(0,T;L2(M)2)|u|4\, L4, |ZM\nabla\theta\cdot|u|2udM|\ \le$
$cZM|\theta||\nabla u||u|2\, dM\ \le\ c|\theta|L4|u|L4||u||\nabla u||L2 \le \nu 1\ 4\ ||u||\nabla u||2\, L2 +$
$c|\theta|2\, L4\ + |\theta|2\, L4|u|4\, L4,$

$|\theta 3\, z|L4\ =\ |\theta 2\, z|3/2\, L6 \le (|\theta 2\, z|L2 + |\nabla 3\theta 2\, z|L2) \le$
$|\theta z|3\, L4 +$ _x0010_ $ZM\theta 2\, z|\nabla\theta z|2dM + ZM\theta 2\, z\theta 2\, zzdM3|\theta 3\, z|L2\ =\ |\theta 2\, z|3/$
$2\, L3\ \le$
c_x0010_$|\theta z|3\, L4 + ZM\theta 2\, z|\nabla\theta z|2dM + ZM\theta 2\, z|\ \partial\theta z\, \partial z\ |2dM\ L2(\Gamma u)\ =$
$|\theta 2|3/2\, L3(\Gamma u) \le\ c$_x0010_$|\theta 2|2/3\, L3(\Gamma u)|\theta 2|1/3\, H1(\Gamma u)3/2\ \le$
_x0010_$|\theta|4/3\, L4(\Gamma u)(|\theta|2/3\, L4(\Gamma u) + |\theta|\nabla\theta||1/3\, L2(\Gamma u))3/2\ \le\ L4(\Gamma u) +$
$|\theta|2\, L4(\Gamma u)|\theta|\nabla\theta||1/2\, L2(\Gamma u). |ZM(vz\ \cdot\nabla)\theta\theta 3\, zdM|\ \le$
$|vz|L4|\nabla\theta|L2|\theta 3\, z|L4 \le$
$|vz|L4|\nabla\theta|Lz|3\, L4 + ZM\theta 2\, z|\nabla\theta z|2dM + ZM\theta 2\, z\theta 2\, zzdM(\nabla\cdot v)\theta 4\, zdM|\ \le$
$|\nabla v|L2|\theta z|L4|\theta 3\, z|L4\ \le\ |\nabla v|L2|\theta z|L4$_x0010_$|\theta z|3\, L4 + (ZM\theta 2\, z|\nabla\theta z|2dM +$
$ZM\theta 2\, z\theta 2\, , |ZM\ \partial F2\, \partial z\ \theta 3\, zdM| \le |\partial F2\, \partial z\ |L2|\theta 3\, z|L2\ \le$
$c|\ \partial F2\, \partial z\ |L2$_x0010_$|\theta z|3\ + ZM\theta 2\, z|\nabla\theta z|2dM +$
$ZM\theta 2\, z\theta 2\, zzdM3/4, |Z\Gamma u\ F2\theta 3dM0| \le$
$c|F2|L2(\Gamma u)\theta|3\, L4(\Gamma u) + |\theta|2\, L4(\Gamma u)|\theta|\nabla\theta||1/2\, L2(M), \le$
$c|F2|L2(\Gamma u)|\theta|4\, L4(\Gamma u) + \nu 2|\theta|\nabla|\theta||2\, L2(\Gamma u)\ +\ c|F2|2\, L2(\Gamma u) +$
$c|F2|L2(\Gamma u).$

we reach
$d\ dt|u|4\, L4 + 2\nu 1||u||\nabla u||2\, L2 + \nu 1|\nabla|u|2|2\, L2 + 2\mu 1||u||\partial u\, \partial z||2\, L2 +$
$\mu 1|\ \partial\, \partial z|u|2|2\, L2 \le$
$c(|v|2\, L6 + |v|4\, L6 + |\theta|2\, L4 + |\partial F1\, \partial z\ |2\, L\infty(0,T;L2(M)2))|u|4\, L4 +$
$c|\theta|2\, L4 + c|\partial F1\, \partial z\ |L\infty(0,T;L2(M)2).$ (3.52)
Gronwall inequality, we have

$$|\partial v\ \partial z|4L4 + Z\ t\ 0\ \nu 1||\partial v\ \partial z||\nabla\partial v\ \partial z|| + \nu 1|\nabla|\partial v\ \partial z|2|2\ L2ds$$

$$+Zt0\ \mu 1||\partial v\ \partial z||\partial 2v\ \partial 2z||2\ L2\ \mu 1|\partial\ \partial z(|\partial v\ \partial z|2)|2\ L2ds \le J8(t),$$

Where

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$J8(t) = ect + c(J1/6\ 1\ J4/3\ 4 + J2\ 1 + J3/2\ 3\ )t\_x0012\_|\partial v0\ \partial z\ |4\ L4(M)$
$+ c(J2\ 2 + |\ \partial F1\ \partial z\ |4\ L\infty$
Step 8. L4 ESTIMATE FOR ∂θ/∂z Taking L2 inner product of the z−derivative of (θz)3 we obtain
1 4
$d\ dt|\theta z|4\ L4 + 3v2ZM\theta2\ z|\nabla\theta z|2dM + 3\mu2ZM\theta2\ z\theta2\ zzdM$
$= -ZM(vz\ \cdot \nabla)\theta\theta3\ zdM + ZM(\nabla \cdot v)\theta4\ zdM$ (3.54) $+ ZM\ \partial F2\ \partial z\theta3\ zdM +$
$\mu2Z\Gamma u\ \theta zz\theta3\ zdM0.$

Using the boundary situation,

$|\theta3\ z|L4 = |\theta2\ z|3/2\ L6 \leq (|\theta2\ z|L2 + |\nabla3\theta2\ z|L2) \leq$
$|\theta z|3\ L4 + \_x0010\_ZM\theta2\ z|\nabla\theta z|2dM + ZM\theta2\ z\theta2\ zzdM3|\theta3\ z|L2 = |\theta2\ z|3/$
$2\ L3\ \leq$
$c\_x0010\_|\theta z|3\ L4 + ZM\theta2\ z|\nabla\theta z|2dM + ZM\theta2\ z|\ \partial\theta z\ \partial z\ |2dM\ L2(\Gamma u)\ =$
$|\theta2|3/2\ L3(\Gamma u) \leq\ c\_x0010\_|\theta2|2/3\ L3(\Gamma u)|\theta2|1/3\ H1(\Gamma u)3/2\ \leq$
$\_x0010\_|\theta|4/3\ L4(\Gamma u)(|\theta|2/3\ L4(\Gamma u)\ + |\theta|\nabla\theta||1/3\ L2(\Gamma u))3/2\ \leq\ L4(\Gamma u) +$
$|\theta|2\ L4(\Gamma u)|\theta|\nabla\theta||1/2\ L2(\Gamma u).$
$|ZM(vz \cdot \nabla)\theta\theta3\ zdM| \leq |vz|L4|\nabla\theta|L2|\theta3\ z|L4 \leq$
$|vz|L4|\nabla\theta|Lz|3\ L4 + ZM\theta2\ z|\nabla\theta z|2dM + ZM\theta2\ z\theta2\ zzdM(\nabla \cdot v)\theta4\ zdM| \leq$
$|\nabla v|L2|\theta z|L4|\theta3\ z|L4 \leq |\nabla v|L2|\theta z|L4\_x0010\_|\theta z|3\ L4\ +\ (ZM\theta2\ z|\nabla\theta z|2dM +$
$ZM\theta2\ z\theta2\ , |ZM\ \partial F2\ \partial z\ \theta3\ zdM|\ \leq\ |\partial F2\ \partial z\ |L2|\theta3\ z|L2 \leq$
$c|\ \partial F2\ \partial z\ |L2\_x0010\_|\theta z|3\ + ZM\theta2\ z|\nabla\theta z|2dM +$
$ZM\theta2\ z\theta2\ zzdM3/4, |Z\Gamma u\ F2\theta3dM0| \leq$
$c|F2|L2(\Gamma u)\theta|3\ L4(\Gamma u) + |\theta|2\ L4(\Gamma u)|\theta|\nabla\theta||1/2\ L2(M), \leq$
$c|F2|L2(\Gamma u)|\theta|4\ L4(\Gamma u) +\ v2|\theta|\nabla|\theta||2\ L2(\Gamma u) + c|F2|2\ L2(\Gamma u) +$
$c|F2|L2(\Gamma u).$

$$\theta z(z\ =\ 0)\ =\ -\alpha\theta(z\ =\ 0),$$

and remaining the equation on the upper boundary Γu, we obtain that
$\mu2Z\Gamma u\ \theta zz\theta3\ zdM0\ = -\alpha Z\Gamma u\_x0010\_\partial\theta\ \partial t\ \theta3\ - v24\theta\theta3\ - F2\theta3dM0\ =$
$-\alpha3\_x0010\_1\ 4\ d\ dt|\theta|4\ L4(\Gamma u)\ +\ 3v2Z\Gamma u\ |\nabla\theta|2\theta2dM0 - Z\Gamma u\ F2\theta3dM0.$

Next, noting that:

$|\theta3\ z|L4 = |\theta2\ z|3/2\ L6\ \leq\ (|\theta2\ z|L2 + |\nabla3\theta2\ z|L2) \leq$
$|\theta z|3\ L4\ + \_x0010\_ZM\theta2\ z|\nabla\theta z|2dM + ZM\theta2\ z\theta2\ zzdM3|\theta3\ z|L2\ =\ |\theta2\ z|3/$
$2\ L3\ \leq$
$c\_x0010\_|\theta z|3\ L4\ + ZM\theta2\ z|\nabla\theta z|2dM + ZM\theta2\ z|\ \partial\theta z\ \partial z\ |2dM\ L2(\Gamma u)\ =$
$|\theta2|3/2\ L3(\Gamma u) \leq\ c\_x0010\_|\theta2|2/3\ L3(\Gamma u)|\theta2|1/3\ H1(\Gamma u)3/2\ \leq$
$\_x0010\_|\theta|4/3\ L4(\Gamma u)(|\theta|2/3\ L4(\Gamma u)\ + |\theta|\nabla\theta||1/3\ L2(\Gamma u))3/2\ \leq\ L4(\Gamma u) +$
$|\theta|2\ L4(\Gamma u)|\theta|\nabla\theta||1/2\ L2(\Gamma u).$

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$|ZM(vz \cdot \nabla)\theta\theta 3\ zdM| \leq |vz|L4|\nabla\theta|L2|\theta 3\ z|L4 \leq\ |vz|L4|\nabla\theta|Lz|3\ L4 +$
$ZM\theta 2\ z|\nabla\theta z|2dM + ZM\theta 2\ z\theta 2\ zzdM(\nabla \cdot v)\theta 4\ zdM| \leq\ |\nabla v|L2|\theta z|L4|\theta 3\ z|L4 \leq$
$|\nabla v|L2|\theta z|L4\_x0010\_|\theta z|3\ L4\ +\ (ZM\theta 2\ z|\nabla\theta z|2dM +$
$ZM\theta 2\ z\theta 2\ , |ZM\ \partial F2\ \partial z\ \theta 3\ zdM|\ \leq\ |\partial F2\ \partial z\ |L2|\theta 3\ z|L2 \leq$
$c|\ \partial F2\ \partial z\ |L2\_x0010\_|\theta z|3 + ZM\theta 2\ z|\nabla\theta z|2dM +$
$ZM\theta 2\ z\theta 2\ zzdM3/4, |Z\Gamma u\ F2\theta 3dM0| \leq$
$c|F2|L2(\Gamma u)\theta|3\ L4(\Gamma u)\ + |\theta|2\ L4(\Gamma u)|\theta|\nabla\theta||1/2\ L2(M), \leq$
$c|F2|L2(\Gamma u)|\theta|4\ L4(\Gamma u)\ +\ v2|\theta|\nabla|\theta||2\ L2(\Gamma u) + c|F2|2\ L2(\Gamma u) +$
$c|F2|L2(\Gamma u).$

Furthermore, having traced up with the theorem:

$F2|L2(\Gamma u) \leq\ c\_x0010\_|F2|L2(M) + |\partial F2\ \partial z\ |L2(M). Hence, |Z\Gamma u\ F2\theta 3dM0| \leq$
$c(|F2|L2 + |\partial F2\ \partial z\ |L2)|\theta|4\ L4(\Gamma u) + v2|\theta|\nabla\theta||2\ L2(\Gamma u) + c|F2|2\ L2\ +$
$c|\partial F2\ \partial z\ |2\ L2 + c|F2|L2 + c|\ \partial F2\ \partial z\ |L2.$
$|\theta 3\ z|L4 = |\theta 2\ z|3/2\ L6 \leq (|\theta 2\ z|L2 + |\nabla 3\theta 2\ z|L2) \leq$
$|\theta z|3\ L4\ + \_x0010\_ZM\theta 2\ z|\nabla\theta z|2dM + ZM\theta 2\ z\theta 2\ zzdM3|\theta 3\ z|L2 = |\theta 2\ z|3/$
$2\ L3\ \leq$
$c\_x0010\_|\theta z|3\ L4 + ZM\theta 2\ z|\nabla\theta z|2dM + ZM\theta 2\ z|\ \partial\theta z\ \partial z\ |2dM\ L2(\Gamma u)\ =$
$|\theta 2|3/2\ L3(\Gamma u) \leq\ c\_x0010\_|\theta 2|2/3\ L3(\Gamma u)|\theta 2|1/3\ H1(\Gamma u)3/2 \leq$
$\_x0010\_|\theta|4/3\ L4(\Gamma u)(|\theta|2/3\ L4(\Gamma u) + |\theta|\nabla\theta||1/3\ L2(\Gamma u))3/2 \leq\ L4(\Gamma u) +$
$|\theta|2\ L4(\Gamma u)|\theta|\nabla\theta||1/2\ L2(\Gamma u).$

$|ZM(vz \cdot \nabla)\theta\theta 3\ zdM| \leq |vz|L4|\nabla\theta|L2|\theta 3\ z|L4 \leq$
$|vz|L4|\nabla\theta|Lz|3\ L4 + ZM\theta 2\ z|\nabla\theta z|2dM + ZM\theta 2\ z\theta 2\ zzdM(\nabla \cdot v)\theta 4\ zdM| \leq$
$|\nabla v|L2|\theta z|L4|\theta 3\ z|L4 \leq |\nabla v|L2|\theta z|L4\_x0010\_|\theta z|3\ L4 +\ (ZM\theta 2\ z|\nabla\theta z|2dM +$
$ZM\theta 2\ z\theta 2\ , |ZM\ \partial F2\ \partial z\ \theta 3\ zdM| \leq\ |\partial F2\ \partial z\ |L2|\theta 3\ z|L2 \leq$
$c|\ \partial F2\ \partial z\ |L2\_x0010\_|\theta z|3 + ZM\theta 2\ z|\nabla\theta z|2dM +$
$ZM\theta 2\ z\theta 2\ zzdM3/4, |Z\Gamma u\ F2\theta 3dM0| \leq$
$c|F2|L2(\Gamma u)\theta|3\ L4(\Gamma u) + |\theta|2\ L4(\Gamma u)|\theta|\nabla\theta||1/2\ L2(M), \leq$
$c|F2|L2(\Gamma u)|\theta|4\ L4(\Gamma u)\ +\ v2|\theta|\nabla|\theta||2\ L2(\Gamma u)$

$$F = (F1, F2) \in L\infty(0, \infty; (L2(M))3), \partial F\infty(0, \infty; (L2(M))3), (v0, \theta 0) \in V\ , \partial v0/\partial z\ \in (L4(M))2\ and\ \partial\theta 0/\partial z \in L4(M).$$

Then basically the 2 γ∗ 1 and γ∗ 2 such that for (v,θ), the strong solution to with the initial condition (vo,θo),
if (vo,θo) and F = (F1,F2) satisfies $|v0|2\ H1 + |\theta 0|2\ H1 + |\ \partial v0\ \partial z\ |2\ L4\ +$
$|\partial\theta 0\ \partial z\ |L4(M) \leq \gamma 1 \leq \gamma * 1,$
$|F|2\ L\infty(0, \infty; (L2(M))3) + |\partial F\partial z\ |2\ L\infty(0, \infty; (L2(M))3) \leq \gamma 2\ \leq\ \gamma * 2,$
then we have
sup $t \geq 0|v(t)|2\ H1 + |\theta(t)|2\ H1 + |\partial v(t)\ \partial z\ |2\ L4 + |\partial\theta(t)\ \partial z\ |2\ L4 \leq$
$C(\gamma 1, \gamma 2),$

when(γ1,γ2) positive numbered property having been determined

γ_1 and γ_2 such that $C(\gamma_1,\gamma_2)$ & 0 as $\gamma_1 + \gamma_2$ & 0.
Here, the constants $\gamma*$ 1 and $\gamma*$ 2 depend only on μ_1, ν_2, and the size of domain M. Assuming $(v(t),\theta(t))$ is the strong solution of with the initial condition (v_0,θ_0) that satisfication. Since $J_1(t)$ and $J_5(t)$–$J_9(t)$ are continuously in state of providing with in temporary constant and agree with the squares of norms of the initial data at t = 0, we see that there exists $t* > 0$ such that $|v(t)|2\ H1 +|\theta(t)|2\ H1 +|\ \partial v(t)\ \partial z\ |2\ L4 +|\ \partial\theta(t)\ \partial z\ |2\ L4 \leq 2(\gamma_1 + \gamma_2), \forall 0 \leq t \leq t*$.

let r be an integral value positive number satisfying $0 < 6r \leq t*$. When it is needed, we may reduce γ_1, γ_2 and r. We shall obtain the smallness condition (3.8) in the following three steps. L2 smallness By (3.11), we have
$|\theta(t)|2\ L2 \leq |\theta 0|2\ L2 + 1\ K2|F2|2\ L\infty(0,\infty; L2(M)) \leq \gamma 1 + 1K2\ \gamma 2 \leq\ C(\gamma 1,\gamma 2)$. (3.63)
Integrating (3.10) with respect to time variable from t to t + r, by (3.63), we obtain $2\nu 2Z\ t + r\ t\ |\nabla\theta|2\ L2\ ds + \mu 2Z\ t + r\ t\ |\ \partial\theta\ \partial z|2\ L2\ ds + \mu 2\alpha Z\ t + r\ t\ |\theta|2\ L2(\Gamma u)\ ds \leq |\theta(t)|2\ L2(M) + 1\ K|F2|2\ L\infty(0,\infty; L2(M)) \leq C(\gamma 1,\gamma 2), \forall t \geq 0$. (3.64)
$|\theta 3\ z|L4 = |\theta 2\ z|3/2\ L6 \leq (|\theta 2\ z|L2 + |\nabla 3\theta 2\ z|L2) \leq$
$|\theta z|3\ L4 + \_x0010\_ZM\theta 2\ z|\nabla\theta z|2dM + ZM\theta 2\ z\theta 2\ zzdM3|\theta 3\ z|L2\ = |\theta 2\ z|3/2\ L3 \leq$
$c\_x0010\_|\theta z|3\ L4 + ZM\theta 2\ z|\nabla\theta z|2dM + ZM\theta 2\ z|\ \partial\theta z\ \partial z\ |2dM\ L2(\Gamma u)\ = |\theta 2|3/2\ L3(\Gamma u) \leq\ c\_x0010\_|\theta 2|2/3\ L3(\Gamma u)|\theta 2|1/3\ H1(\Gamma u)3/2 \leq$
$\_x0010\_|\theta|4/3\ L4(\Gamma u)(|\theta|2/3\ L4(\Gamma u) + |\theta|\nabla\theta||1/3\ L2(\Gamma u))3/2 \leq\ L4(\Gamma u) + |\theta|2\ L4(\Gamma u)|\theta|\nabla\theta||1/2\ L2(\Gamma u)$.
$|ZM(vz\cdot\nabla)\theta\theta 3\ zdM| \leq |vz|L4|\nabla\theta|L2|\theta 3\ z|L4 \leq$
$|vz|L4|\nabla\theta|Lz|3\ L4 + ZM\theta 2\ z|\nabla\theta z|2dM + ZM\theta 2\ z\theta 2\ zzdM(\nabla\cdot v)\theta 4\ zdM| \leq$
$|\nabla v|L2|\theta z|L4|\theta 3\ z|L4 \leq |\nabla v|L2|\theta z|L4\_x0010\_|\theta z|3\ L4\ +\ (ZM\theta 2\ z|\nabla\theta z|2dM + ZM\theta 2\ z\theta 2\ ,|ZM\ \partial F2\ \partial z\ \theta 3\ zdM| \leq\ |\partial F2\ \partial z\ |L2|\theta 3\ z|L2 \leq$
$c|\ \partial F2\ \partial z\ |L2\_x0010\_|\theta z|3 + ZM\theta 2\ z|\nabla\theta z|2dM +$
$ZM\theta 2\ z\theta 2\ zzdM3/4, |Z\Gamma u\ F2\theta 3dM0| \leq$
$c|F2|L2(\Gamma u)\theta|3\ L4(\Gamma u) + |\theta|2\ L4(\Gamma u)|\theta|\nabla\theta||1/2\ L2(M), \leq$
$c|F2|L2(\Gamma u)|\theta|4\ L4(\Gamma u)$

we having
$|v(t)|2\ L2 \leq C(\gamma_1,\gamma_2), \forall t \geq 0$.
Step 2. H1 smallness Integrating with respect to time variable over [t,t + r], we obtain $\nu 1Z\ t + r\ t\ |\nabla v|2\ L2\ ds + 2\mu 1Z\ t + r\ t\ |\ \partial v\ \partial z|2\ L2\ ds \leq\ C(\gamma 1,\gamma 2), \forall t\ \geq 0$.
As for the L6 estimate of θ, we note that
$|\theta_0|2\ L6 \leq c|\theta_0|2\ H1 \leq c\gamma_1$
Hence, by we obtain $|\theta(t)|2\ L6 \leq C(\gamma_1,\gamma_2), \forall t \geq 0$. (3.68) By and the assumption (3.59), we see that $B_1(t)$ in satisfies $B_1(t) \leq C(\gamma_1,\gamma_2) + c|\nabla v|2\ L2$.

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Hence, by (3.66), we obtain $Z\ t + r\ t\ B1(s)ds \leq C(\gamma1,\gamma2), t \geq 0$. (3.69) Next, by (3.67) and the assumption (3.59), we see that B2(t) in satisfies Z t+r t B2(s)ds ≤ C(γ1,γ2), t ≥ 0.Notice that $|\tilde{}\ v|2\ L6 \leq c(|\bar{}v|2\ L6(M0) + |v|2\ L6) \leq c|v|L6 \leq |\nabla v|2\ L2$.
We then derive from (3.66) that

$$Z\ t + r\ t\ |\tilde{}\ v|2\ L6\ ds \leq cZ\ t + r\ t\ |\nabla v|2\ L2\ ds \leq C(\gamma1,\gamma2), \forall t \geq 0.$$

$|\theta3\ z|L4 = |\theta2\ z|3/2\ L6 \leq (|\theta2\ z|L2 + |\nabla3\theta2\ z|L2) \leq$
$|\theta z|3\ L4 + \_x0010\_ZM\theta2\ z|\nabla\theta z|2dM + ZM\theta2\ z\theta2\ zzdM3|\theta3\ z|L2\ = |\theta2\ z|3/2\ L3\ \leq$
$c\_x0010\_|\theta z|3\ L4 + ZM\theta2\ z|\nabla\theta z|2dM + ZM\theta2\ z|\ \partial\theta z\ \partial z\ |2dM\ L2(\Gamma u)\ =$
$|\theta2|3/2\ L3(\Gamma u) \leq\ c\_x0010\_|\theta2|2/3\ L3(\Gamma u)|\theta2|1/3\ H1(\Gamma u)3/2\ \leq$
$\_x0010\_|\theta|4/3\ L4(\Gamma u)(|\theta|2/3\ L4(\Gamma u) + |\theta|\nabla\theta||1/3\ L2(\Gamma u))3/2\ \leq\ L4(\Gamma u) +$
$|\theta|2\ L4(\Gamma u)|\theta|\nabla\theta||1/2\ L2(\Gamma u)$.
$|ZM(vz\cdot\nabla)\theta\theta3\ zdM| \leq |vz|L4|\nabla\theta|L2|\theta3\ z|L4 \leq$
$|vz|L4|\nabla\theta|Lz|3\ L4 + ZM\theta2\ z|\nabla\theta z|2dM + ZM\theta2\ z\theta2\ zzdM(\nabla\cdot v)\theta4\ zdM| \leq$
$|\nabla v|L2|\theta z|L4|\theta3\ z|L4 \leq |\nabla v|L2|\theta z|L4\_x0010\_|\theta z|3\ L4 +\ (ZM\theta2\ z|\nabla\theta z|2dM +$
$ZM\theta2\ z\theta2\ , |ZM\ \partial F2\ \partial z\ \theta3\ zdM|\ \leq\ |\partial F2\ \partial z\ |L2|\theta3\ z|L2 \leq$
$c|\ \partial F2\ \partial z\ |L2\_x0010\_|\theta z|3\ + ZM\theta2\ z|\nabla\theta z|2dM +$
$ZM\theta2\ z\theta2\ zzdM3/4, |Z\Gamma u\ F2\theta3dM0| \leq$
$c|F2|L2(\Gamma u)\theta|3\ L4(\Gamma u)\ + |\theta|2\ L4(\Gamma u)|\theta|\nabla\theta||1/2\ L2(M), \leq$
$c|F2|L2(\Gamma u)|\theta|4\ L4(\Gamma u)\ +\ \nu2|\theta|\nabla|\theta||2\ L2(\Gamma u) + c|F2|2\ L2(\Gamma u) + c|F2|L2(\Gamma u)$
Inferring from (3.39), by (3.69) –(3.71) and the uniform l lemma, we obtain
$|\tilde{}\ v(t)|2\ L6 \leq C(\gamma1,\gamma2), \forall t\ \geq\ r$. (3.72)
Moreover, by we derive from that
$Z\ t + r\ t\ ||\tilde{}\ v|2|\nabla\tilde{}\ v||2\ L2\ ds \leq C(\gamma1,\gamma2), \forall t\ \geq\ 2r$.
the uniform lemma, we derive from that
$|\nabla\bar{}\ v|2\ L2(M0)\ \leq\ C(\gamma1,\gamma2), \forall t\ \geq\ 3r$.
Meanwhile, we obtain $|v|L6\ \leq\ C(\gamma1,\gamma2), \forall t\ \geq\ 3r$.
Again, by the uniform Gronwall lemma, and, we derive from that $|\ \partial v\ \partial z|L2\ \leq\ C(\gamma1,\gamma2), \forall t\ \geq\ 4r$ .Moreover, it is inferred from that $Z\ t + r\ t\ \nu1|\nabla\partial v\ \partial z|2\ L2\ +\ \mu1|\ \partial2v\ \partial z2|2\ L2ds\ \leq\ C(\gamma1,\gamma2), \forall t\ \geq\ 4r$. Inferring from (3.48), by and the uniform Gronwall lemma, we have $|\nabla v|L2 \leq C(\gamma1,\gamma2), \forall t\ \geq\ 5r$.
Moreover, from, by direct integration, we have $Z\ t + r\ t\ |4v|2\ L2\ dM \leq C(\gamma1,\gamma2), \forall t\ \geq\ 5r$. Applying the uniform l lemma, we have

$$|\nabla\theta|2\ L2 + |\ \partial\theta\ \partial z|2\ L2 + \alpha|\nabla\theta|2\ L2(\Gamma u) \leq C(\gamma1,\gamma2), \forall t \geq 6r$$
(3.80)

Step 3. L4 smallness. To manipulate the L4 smallness of ∂v/∂z, we have the following observations. First, due to the boundary condition ∂v ∂z = 0, on Γb, the Poincare inequality for |vz|2 gives the inequality
$|vz|4\ L4\ \leq ||vz|2|2\ L2\ \leq\ c|\ \partial\ \partial z\ (|vz|2)|L2$. (3.81)
We may choose γ1 and γ2 small so that, by (3.75) and (3.68), we have

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$c(|v|2\ L6 + |v|4\ L6 + |\theta|2\ L4 + |\ \partial F1\ \partial z\ |2\ L\infty(0,T; L2(M)2))|vz|4\ L4\ \leq$
$C(\gamma 1, \gamma 2)|\ \partial\ \partial z\ (|vz|2)|2\ L2 \leq$ µ1 2 $|\ \partial z\ (|vz|2)|2\ L2, \forall t\ \geq\ 3r.$ (3.82)
$|\theta 3\ z|L4 = |\theta 2\ z|3/2\ L6 \leq (|\theta 2\ z|L2 + |\nabla 3\theta 2\ z|L2) \leq$
$|\theta z|3\ L4\ + \_x0010\_ZM\theta 2\ z|\nabla\theta z|2dM + ZM\theta 2\ z\theta 2\ zzdM3|\theta 3\ z|L2 = |\theta 2\ z|3/$
$2\ L3\ \leq$
$c\_x0010\_|\theta z|3\ L4 + ZM\theta 2\ z|\nabla\theta z|2dM + ZM\theta 2\ z|\ \partial\theta z\ \partial z\ |2dM\ L2(\Gamma u)\ =$
$|\theta 2|3/2\ L3(\Gamma u) \leq\ c\_x0010\_|\theta 2|2/3\ L3(\Gamma u)|\theta 2|1/3\ H1(\Gamma u)3/2\ \leq$
$\_x0010\_|\theta|4/3\ L4(\Gamma u)(|\theta|2/3\ L4(\Gamma u)\ + |\theta|\nabla\theta||1/3\ L2(\Gamma u))3/2\ \leq\ L4(\Gamma u) +$
$|\theta|2\ L4(\Gamma u)|\theta|\nabla\theta||1/2\ L2(\Gamma u).$

$|ZM(vz\ \cdot \nabla)\theta\theta 3\ zdM| \leq |vz|L4|\nabla\theta|L2|\theta 3\ z|L4 \leq |vz|L4|\nabla\theta|Lz|3\ L4\ +$
$ZM\theta 2\ z|\nabla\theta z|2dM + ZM\theta 2\ z\theta 2\ zzdM(\nabla \cdot v)\theta 4\ zdM|\ \leq\ |\nabla v|L2|\theta z|L4|\theta 3\ z|L4 \leq$
$|\nabla v|L2|\theta z|L4\_x0010\_|\theta z|3\ L4 +\ (ZM\theta 2\ z|\nabla\theta z|2dM +$
$ZM\theta 2\ z\theta 2\ , |ZM\ \partial F2\ \partial z\ \theta 3\ zdM|\ \leq\ |\partial F2\ \partial z\ |L2|\theta 3\ z|L2 \leq$
$c|\ \partial F2\ \partial z\ |L2\_x0010\_|\theta z|3\ + ZM\theta 2\ z|\nabla\theta z|2dM +$
$ZM\theta 2\ z\theta 2\ zzdM3/4, |Z\Gamma u\ F2\theta 3dM0|\ \leq$
$c|F2|L2(\Gamma u)\theta|3\ L4(\Gamma u) + |\theta|2\ L4(\Gamma u)|\theta|\nabla\theta||1/2\ L2(M), \leq$
$c|F2|L2(\Gamma u)|\theta|4\ L4(\Gamma u) + v2|\theta|\nabla|\theta||2\ L2(\Gamma u) + c|F2|2\ L2(\Gamma u)\ +$
$c|F2|L2(\Gamma u)$

By (3.82) and Gronwall inequality, we infer from (3.52) that
$|\ \partial v\ \partial z|2\ L4\ \leq\ C(\gamma 1, \gamma 2), \forall t\ \geq\ 3r.$ (3.83)
To proceed the L4 smallness of ∂θ/∂z, we notice that the boundary conditions ∂θ ∂z = 0 on Γb, θ = 0 on Γl, give the Poincare inequalities

$|\theta 2\ z|2\ L2 \leq cZM\theta 2\ z\theta 2\ zzdM, |\theta 2|2\ L2(\Gamma u) \leq cZ\Gamma u\ \theta 2|\nabla\theta|2dM0.$

Now, choose γ∗ 1 and γ∗ 2 small enough so that for γ1 ≤ γ∗ 1, γ2 ≤ γ∗ 2, (3.84) we have not only (3.82) but also
$c(|\nabla\theta|4/3\ L2 + |\nabla v|L2 + |\nabla v|4\ L2 + |F2|L2 + |\ \partial F2\ \partial z\ |L2)(|\theta z|4\ L4 +$
$\alpha 3|\theta|4\ L4(\Gamma u)) \leq 1\ 2\ v2\_x0010\_ZM\theta 2\ z|\nabla\theta z|2dM +\ \alpha 3Z\Gamma u\ \theta 2|\nabla\theta|2dM0.$
(3.85) Hence, by (3.80), (3.78), (3.83) and Gronwall inequality, inferring from (3.57),

we obtain
$|\theta z|4\ L4 + \alpha 3|\theta|4\ \Gamma u \leq C(\gamma 1, \gamma 2), \forall t\ \geq\ 5r,$ (3.86) provided γ1 and γ2 are small. Combining (3.63), (3.65), (3.80), (3.78), (3.76) and (3.83), we have
$|v|2\ H1 + |\ \partial v\ \partial z|2\ L4 + |\theta|2\ H1 + |\ \partial\theta\ \partial z|2\ L4 \leq C(\gamma 1, \gamma 2), \forall t\ \geq\ 6r.$ (3.87)
To sum up, by (3.62) and (3.87),

$\sup\ t \geq 0|v|2\ H1 + |\theta|2\ H1 + |\partial v(t)\ \partial z\ |2\ L4 + |\partial\theta(t)\ \partial z\ |2\ L4 \leq\ C(\gamma 1, \gamma 2).$

The proof of the lemma is complete.

4. Main Theorem and its proof

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In this section, we state our main theorems and give the complete proofs. The first main result regarding the asymptotic stability issue is as follows. Theorem 4.1. Suppose F = (F1,F2),∂F/∂z ∈ L∞(0,∞;(L2(M))3). There exists a positive number ˜ γ2 such that if |F|2 L∞(0,∞;(L2(M))3) +| ∂F ∂z |2 L∞(0,∞;(L2(M))3) ≤ ˜ γ2, (4.1) then for any two strong solutions (v1(t),θ1(t)) and (v2(t),θ2(t)) of the system, we have lim t→∞_x0010_|v1(t)−v2(t)|2 L2 +|θ1(t)−θ2(t)|2 L2= 0.

|𝜃3 𝑧|𝐿4 = |𝜃2 𝑧|3/2 𝐿6 ≤ (|𝜃2 𝑧|𝐿2 + |𝛻3𝜃2 𝑧|𝐿2) ≤
|𝜃𝑧|3 𝐿4 + _𝑥0010_𝑍𝑀𝜃2 𝑧|𝛻𝜃𝑧|2𝑑𝑀 + 𝑍𝑀𝜃2 𝑧𝜃2 𝑧𝑧𝑑𝑀3|𝜃3 𝑧|𝐿2 = |𝜃2 𝑧|3/
2 𝐿3 ≤
𝑐_𝑥0010_|𝜃𝑧|3 𝐿4 + 𝑍𝑀𝜃2 𝑧|𝛻𝜃𝑧|2𝑑𝑀 + 𝑍𝑀𝜃2 𝑧| ∂𝜃𝑧 ∂𝑧 |2𝑑𝑀 𝐿2(𝛤𝑢) =
|𝜃2|3/2 𝐿3(𝛤𝑢) ≤ 𝑐_𝑥0010_|𝜃2|2/3 𝐿3(𝛤𝑢)|𝜃2|1/3 𝐻1(𝛤𝑢)3/2 ≤
_𝑥0010_|𝜃|4/3 𝐿4(𝛤𝑢)(|𝜃|2/3 𝐿4(𝛤𝑢) + |𝜃|𝛻𝜃||1/3 𝐿2(𝛤𝑢))3/2 ≤ 𝐿4(𝛤𝑢) +
|𝜃|2 𝐿4(𝛤𝑢)|𝜃|𝛻𝜃||1/2 𝐿2(𝛤𝑢).

|𝑍𝑀(𝑣𝑧 · 𝛻)𝜃𝜃3 𝑧𝑑𝑀| ≤ |𝑣𝑧|𝐿4|𝛻𝜃|𝐿2|𝜃3 𝑧|𝐿4 ≤ |𝑣𝑧|𝐿4|𝛻𝜃|𝐿𝑧|3 𝐿4 +
𝑍𝑀𝜃2 𝑧|𝛻𝜃𝑧|2𝑑𝑀 + 𝑍𝑀𝜃2 𝑧𝜃2 𝑧𝑧𝑑𝑀(𝛻 · 𝑣)𝜃4 𝑧𝑑𝑀| ≤ |𝛻𝑣|𝐿2|𝜃𝑧|𝐿4|𝜃3 𝑧|𝐿4 ≤
|𝛻𝑣|𝐿2|𝜃𝑧|𝐿4_𝑥0010_|𝜃𝑧|3 𝐿4 + (𝑍𝑀𝜃2 𝑧|𝛻𝜃𝑧|2𝑑𝑀 +
𝑍𝑀𝜃2 𝑧𝜃2 , |𝑍𝑀 ∂𝐹2 ∂𝑧 𝜃3 𝑧𝑑𝑀| ≤ |∂𝐹2 ∂𝑧 |𝐿2|𝜃3 𝑧|𝐿2 ≤
𝑐| ∂𝐹2 ∂𝑧 |𝐿2_𝑥0010_|𝜃𝑧|3 + 𝑍𝑀𝜃2 𝑧|𝛻𝜃𝑧|2𝑑𝑀 +
𝑍𝑀𝜃2 𝑧𝜃2 𝑧𝑧𝑑𝑀3/4, |𝑍𝛤𝑢 𝐹2𝜃3𝑑𝑀0| ≤
𝑐|𝐹2|𝐿2(𝛤𝑢)𝜃|3 𝐿4(𝛤𝑢)|𝜃|2 𝐿4(𝛤𝑢)|𝜃|𝛻𝜃||1/2 𝐿2(𝑀), ≤
𝑐|𝐹2|𝐿2(𝛤𝑢)|𝜃|4 𝐿4(𝛤𝑢) + 𝑣2|𝜃|𝛻|𝜃||2 𝐿2(𝛤𝑢) + 𝑐|𝐹2|2 𝐿2(𝛤𝑢) +
𝑐|𝐹2|𝐿2(𝛤𝑢).

Results and Discussion

A fundamental work for investigating asymptotic soundness of Primitive conditions with time-subordinate driving, we follow the thoughts of and demonstrate the worldwide in time presence of answers for Primitive conditions in our setting. We next exploit the uniform Gronwall lemma to get the appraisals to build up a coupled framework of conventional differential imbalances concerning the energy appraisals of fluid speed and the temperature work. Under a reasonable diminutiveness condition on the constraining term, the asymptotic solidness of arrangements with little beginning information is then gotten from this arrangement of differential imbalances.

Under a comparative setting, in Tachim treated the presence of time-intermittent arrangements of the Primitive conditions by Galerkin's technique under a generally more grounded supposition that the warmth source is differentiable in the transient variable fulfilling some diminutiveness conditions. Notwithstanding, the creator didn't address the security issue in. In this article, we loosen up the routineness prerequisite of driving terms and

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we give an asymptotic strength investigation. The thought we utilize to demonstrate the presence and uniqueness of time occasional solid arrangements depends on a Serrin's strategy, which we think all the more intelligently direct and numerically wonderful. It is worth-referenced that our investigation can be applied to two-dimensional Navier-Stokes conditions combined with heat diffusion conditions on limited areas. It should be simple for intrigued perusers to supply the essential subtleties.
The Primitive conditions and their varieties are that we restate some Sobolev type imbalances, uniform Gronwall disparity and some significant numerical outcomes on Primitive conditions. The worldwide in time presence of the arrangement and a vital boundedness lemma are expressed and demonstrated. At long last, we state and demonstrate our primary hypotheses. Investigation for different spaces, such as the round shell area or unbounded area, should be tended to elsewhere.

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