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Certain Geometric Properties of Four Parametric Wright Functions

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Abstract

In this article, we introduce and examine new integral operators involving four parametric Wright functions. These operators extend and generalize existing integral operators found in the literature. We explore several geometric properties of these new operators, including univalency and convexity. Our discussion focuses on how these properties manifest in the context of four parametric Wright functions.

Keywords: Convex Function, Star like Function, Univalency, Close to Convex Function.

Introduction

The four parametric Wright function

$$W_{(\alpha,\beta,c,d)}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(c+m\alpha)\Gamma(d+m\beta)},c,d\in\mathbb{C},\ \alpha,\beta\in\mathbb{R}. \tag{1.1}$$

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Vol. 3 No. 3 (March) (2025)

was introduced by E. M. Wright for $\alpha,\beta>0$. The series defined in equation (1.1) converges absolutely and it is an entire function [1-3]. The wright function can be seen as a special case of four parametric wright function. The Wright function

$$W_{\varsigma,\eta}(z)=\sum_{l=0}^{\infty}\frac{z^{l}}{l!\Gamma(\varsigma l+\eta)},\varsigma>-1,\eta\in\mathbb{C}, \tag{1.2}$$

was introduced by E. M. Wright [4] in 1933 and has since been applied in various fields, including the partitions of asymptotic theory, also the theory of Hankel type integral transformation and operational calculus of Mikusinski. Wright functions played a role in solving pde's of fractional order with corresponding Green functions being expressed in terms of Wright functions [5, 6]. Mainardi [7] used Wright functions to solve the fractional diffusion wave equation. Luchko et al. [1, 8] derived scale invariant results of pde's (partial differential equations) of fractional order in form of Wright functions. In 2022 M. U. Din [9] determined the partial sums of four- parametric Wright function.

Let Λ be the class having the functions t in the form of

$$t(r)=r+\sum_{l=2}^{\infty}c_{l}r^{l}, \tag{1.3}$$

analytic contained in open unit disc $U=\{r\,|\,|r|<1\}$.

Wright function, especially in its four parametric form, is defined as a series that generalizes several well-known special functions, including the generalized hypergeometric function and the Mittag-Leffler function. Recently, numerous researchers have investigated various geometric belongings, such as univalency, convexity, star likeness as well as close-to-convexity of special functions. Studies have been conducted on geometric properties of hypergeometric functions [10, 11], Bessel functions [12], Struve functions [13], Lommel functions [14].This body of work led Sourav Das and Khaled Mehrez [15] to explore the geometric properties of four parametric Wright function. we will derive some sufficient conditions by utilizing inequalities related to four parametric Wright functions.

We consider the following normalization of $W_{(\mu,c)}^{(v,d)}(z)$

$$W_{(\mu,c)}^{(v,d)}(z)=\sum_{h=0}^{\infty}\frac{\Gamma c\Gamma e}{(\Gamma c+l\mu)(\Gamma e+kv)}z^{h+1},$$

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Vol. 3 No. 3 (March) (2025)

$$= z + \sum_{h=2}^{\infty} \frac{\Gamma c \Gamma e}{\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v)} z^h .$$

We requisite following lemmas to verify our key results:

Lemma 1[16]:

If function $g(z) = z + c_2 z^2 + \ldots + c_n z^n + \ldots$ is analytic in U also

$1 \geq 2c_2 \geq \ldots \geq nc_n \ldots \geq 0$,

Or

$1 \leq 2c_2 \leq \ldots \leq nc_n \ldots \leq 2$

is close-to-convex regarding to convex function $z \to -\log(1-z)$

Lemma 2 [14]:

If function $g(z) = z + d_3 z^3 + \ldots + d_{2n-1} z^{2n-1} + \ldots$ analytic in U also if

$1 \geq 3d_3 \geq \ldots \geq (2n+1) d_{2n+1} \ldots \geq 0$,

Or

$1 \leq 3d_3 \leq \ldots \leq (2n+1) d_{2n+1} \ldots \leq 2$.

Then $g(z)$ is univalent in U.

Main Results

Theorem 2.1: If $c, e, \mu, v \in \mathbb{R}^+$ with inequality

$$h\Gamma(c+h\mu)\Gamma(e+hv) \geq (h+1)\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v),$$

Then

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www.thedssr.com
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ISSN Print: 3007-3146
Vol. 3 No. 3 (March) (2025)

$$\hat{z} \rightarrow W_{(\mu,c)}^{(v,e)}(\hat{z}),$$

is close to convex regarding to convex function $-\log(1-z)$.

Proof:

Consider

$$W_{(\mu,c)}^{(v,e)}(\hat{z}) = \hat{z} + \sum_{k=2}^{\infty} \frac{\Gamma c \Gamma e}{\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v)} \hat{z}^h .$$

Here,

$$\hat{a}_{h-1} = \frac{\Gamma c \Gamma e}{\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v)},$$

We have $\hat{a}_{h-1} > 0$ for all $h \geq 2$.

For $l = 2$,

$$\hat{a}_1 = \frac{\Gamma c \Gamma e}{\Gamma(c+\mu)\Gamma(e+v)}$$

$$< 1.$$

Now, we have

$$\Omega \hat{a}_h = h\hat{a}_{h-1} - (h+1)\hat{a}_h.$$

$$= h\frac{\Gamma c \Gamma e}{\Gamma(c+h-1)\mu)\Gamma(e+h-1)v)} - (h+1)\frac{\Gamma c \Gamma e}{\Gamma(c+h\mu)\Gamma(e+hv)}$$

$$= \Gamma c \Gamma e \left\{ \frac{h}{\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v)} - \frac{(h+1)}{\Gamma(c+h\mu)\Gamma(e+hv)} \right\}$$

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$$=\Gamma c\Gamma e\left\{\frac{h\Gamma(c+h\mu)\Gamma(e+hv)-(h+1)\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v)}{\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v)\Gamma(c+h\mu)\Gamma(e+hv)}\right\}.$$

Here

$$\Gamma c>0,\ \Gamma e>0$$

$$\Gamma(c+(h-1)\mu)>0,$$

$$\Gamma(e+(h-1)v)>0,$$

$$\Gamma(c+h\mu)>0$$

and

$$\Gamma(c+hv)>0,$$

Under the conditions on parameters, for $h\geq 2$. In view of Lemma 2.1, we have to show that $\Omega\hat{a}_h\geq 0$. Here we observe that $\Omega\hat{a}_h$ is positive if,

$$h\Gamma(c+h\mu)\Gamma(e+hv)\geq(h+1)\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v),$$

for all $h\geq 2$.

Thus $W_{(\mu,c)}^{(v,e)}(\hat{z})$ is close to convex regarding to convex function $\left(-\log(1-\hat{z})\right)$.

Theorem 2.2: If $c,e,\mu,v\in\mathbb{R}^+$ with inequality

$$(2h-1)\Gamma(c+h\mu)\Gamma(e+hv)\geq(2h+1)\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v),$$

then

$$\hat{z}\longrightarrow W_{(\mu,c)}^{(v,e)}(\hat{z}^2),$$

is close to convex regarding to convex function $\left(\frac{1}{2}\log\frac{1+\hat{z}}{1-\hat{z}}\right)$.

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Vol. 3 No. 3 (March) (2025)

Proof: Let

$$f(\hat{z}) = \hat{z}W_{(\mu,c)}^{(v,e)}(\hat{z}^2)$$

$$= \hat{z} + \sum_{h=2}^{\infty} A_{2h-1}\hat{z}^{2h-1}.$$

Here

$$A_{2h-1} = \hat{a}_{h-1} = \frac{\Gamma c \Gamma e}{\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v)}.$$

Therefore, we have

$$\hat{a}_1 = \frac{\Gamma c \Gamma e}{\Gamma(c+\mu)\Gamma(e+v)}$$

$$< 1,$$

and $A_{2h-1} > 0$ for all $h \geq 2$.

Firstly we will show that $\left\{(2h-1)e_{h-1}\right\}_{m\geq 2}$ is a sequence of decreasing functions.

For this consider

$$\left(2h-1\right)\hat{a}_{h-1} - \left(2h+1\right)a_h \geq 0$$

$$= \frac{\left(2h-1\right)\Gamma c \Gamma e}{\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v)} - \frac{\left(2h+1\right)\Gamma c \Gamma e}{\Gamma(c+h\mu)\Gamma(e+hv)}.$$

$$= \Gamma c \Gamma e \left\{ \frac{(2h-1)}{\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v)} - \frac{(2h+1)}{\Gamma(c+h\mu)\Gamma(e+hv)} \right\}$$

$$= \Gamma c \Gamma e \left\{ \frac{(2h-1)\Gamma(c+h\mu)\Gamma(e+hv) - (2h+1)\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v)}{\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v)\Gamma(c+h\mu)\Gamma(e+hv)} \right\}.$$

Here

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www.thedssr.com
ISSN Online: 3007-3154
ISSN Print: 3007-3146
Vol. 3 No. 3 (March) (2025)

$$\Gamma c > 0\,,\ \Gamma e > 0$$

$$\Gamma(c+(h-1)\mu) > 0\,,$$

$$\Gamma(e+(h-1)v) > 0\,,$$

$$\Gamma(c+h\mu) > 0$$

And

$$\Gamma(e+hv) > 0\,,$$

Under the conditions on parameters, for $h \geq 2$. For

$$(2h-1)\Gamma(c+h\mu)\Gamma(e+hv) \geq (2h+1)\Gamma(c+(h-1)\mu)\Gamma(e+(h-1)v),$$

we observe that

$$\left(2h-1\right)\hat{a}_{h-1} - \left(2h+1\right)\hat{a}_h \geq 0 \text{ for all } h \geq 2.$$

Hence, the sequence $\left\{\left(2h-1\right)b_{h-1}\right\}_{m\geq 2}$ is decreasing sequence. By using Lemma 2.2, we have $\hat{z}W_{(\mu,c)}^{(v,e)}(\hat{z}^2)$ is close to convex regarding to function $\left(\frac{1}{2}\log\frac{1+\hat{z}}{1-\hat{z}}\right)$.

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