Dialogue Social Science Review (DSSR)
www.thedssr.com

ISSN Online: 3007-3154
ISSN Print: 3007-3146

DIALOGUE SOCIAL SCIENCE REVIEW

Vol. 3 No. 3 (March) (2025)

Geometric Properties of Barnes Mittage-Leffler Functions

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Abstract
In this paper, we explore the geometric properties of the Barnes-Mittage-Leffler functions, focusing particularly on their q-close-to-convex behaviors. By examining the relationships between these functions and their geometric constraints, we aim to provide new insights into their applications in complex analysis, geometric function theory, and mathematical physics. Through this exploration, we also highlight potential future directions for further research, especially in the context of fractional calculus and its connection to the geometry of special functions.

Keywords: Convex Function, Star like Function, Univalency, Close to Convex Function. q-Close-to-Convex Function

Introduction

Consider the class Λ of analytic function, defined as

$$\hbar(\widehat{z}) = \widehat{z} + \sum_{t=2}^{\infty} d_t \widehat{z}^t, \qquad (1.1)$$

in the open unit disc $\widehat{U} = \{\widehat{z} \mid |\widehat{z}| < 1\}$. Let $\widehat{S}$ be the class of univalent as well as analytic functions. The classes of star like and close to convex functions of order $\widehat{\alpha}$ can be represent as:

$$\widehat{S}^*(\widehat{\alpha}) = \left\{ \hbar : \hbar \in \widehat{S}, and \operatorname{Re} \frac{\widehat{z}\hbar'(\widehat{z})}{\hbar(\widehat{z})} > \widehat{\alpha} \right\}.$$

And

$$\kappa_\tau(\widehat{\alpha}) = \left\{ \hbar : \hbar \in \widehat{S}, and \operatorname{Re} \frac{\widehat{z}\hbar'(\widehat{z})}{\tau(\widehat{z})} > \widehat{\alpha}, \tau \in \widehat{S}^* \right\}$$

respectively. Clearly $\widehat{S}^*(0) = \widehat{S}^*$ and $\kappa_\tau(0) = \kappa_\tau$ are well—known classes of star

ISSN Online: 3007-3154
ISSN Print: 3007-3146

Vol. 3 No. 3 (March) (2025)

like and close-to-convex functions respectively.

Notations and Definitions

Now we discuss some basic notations and definitions related to $\hat{q}$ calculus.

For $\hat{q} \in (0,1)$, the $\hat{q}$ number $[\varpi]_{\hat{q}}!$ is

$$[\varpi]_{\hat{q}} = \begin{cases} \dfrac{1-\hat{q}^{\varpi}}{1-\hat{q}}, & \varpi \in \mathcal{C}, \\ \sum\limits_{t=0}^{\varpi-1} \hat{q}^{t}, & \varpi \in N. \end{cases}$$

And, the $\hat{q}$ factorial $[\varpi]_{\hat{q}}!$ is represented as

$$[0]_{\hat{q}}! = 1, \qquad [\varpi]_{\hat{q}}! = \prod_{t=0}^{\varpi} [t]_{\hat{q}}. \qquad \varpi \in N.$$

Consider $\hat{b}$, $\hat{q} \in \mathcal{C}\ (|\hat{q}| < 1)$, $\varpi \in \mathbb{N}_{\circ} = \mathbb{N} \cup \{0\}$. Then the $\hat{q}$-shifted factorial $(\hat{b}\,;\hat{q})_{t\varpi}$ is represented as

$$(\hat{b}\,;\hat{q})_0 = 1, \quad (\hat{b}\,;\hat{q})_{\varpi} = \prod_{t=1}^{\varpi} \left(1 - \hat{b}\,, \hat{q}^{\,t-1}\right), \qquad \varpi \in \mathbb{N}.$$

For χ $\in \mathcal{C} - \{-\varpi : \varpi \in \mathbb{N}_0\}$. Then q-Gamma function is stated by

$$\Gamma_{\hat{q}}(\chi) = \frac{(\hat{q}\,;\hat{q})_{\infty}}{\left(\hat{q}^{\,\chi}\,;\hat{q}\right)_{\infty}} (1-\hat{q})^{1-\chi}, \qquad 0 < \hat{q} < 1.$$

Now the $\hat{q}$-derivative operator $D_{\hat{q}}\iota$ of a function ι is represented by

$$D_{\hat{q}}\iota(\hat{z}) = \begin{cases} \dfrac{\iota(\hat{z}) - \iota(\hat{q}\hat{z})}{\hat{z}\,(1-\hat{q})}, & \hat{z} \neq 0, \\ \iota'(0), & \hat{z} = 0. \end{cases} \qquad (2)$$

$\iota'(0)$ exists.

The class of $\kappa_{\hat{q},g}$ [1]

The class of $\kappa_{\hat{q},g}$ i.e. $\hat{q}$ close-to-convex function is defined by using the $\hat{q}$ derivative operator as:

$$\left|\frac{\hat{z}}{g(\hat{z})}\left(D_{\hat{q}}\iota\right)\hat{z} - \frac{1}{1-\hat{q}}\right| \le \frac{1}{1-\hat{q}'}, \hat{z} \in \Delta, \hat{q} \in (0,1). \qquad (3)$$

The class $\kappa_{\hat{q},g}$ becomes the class κ_{τ} , when $\hat{q} \to 1^{-}$.

Special functions play a crucial role in various branches of mathematics, physics, and engineering due to their ability to describe complex phenomena in a compact and elegant manner. These functions, often arising as solutions to differential equations or integral representations, are indispensable tools for tackling problems in areas such as complex analysis, geometric function theory, and

ISSN Online: 3007-3154
ISSN Print: 3007-3146

DIALOGUE SOCIAL SCIENCE REVIEW

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mathematical physics. In geometric function theory, these functions have made significant contributions, especially in the solution of Bieber Bach conjecture[2]. A vast body of literature explores the geometric properties of various types of special functions [3-5]. For example, Owa and Srivastava [6] investigated the univalence and starlikeness of hypergeometric functions. Srivastava and Dziok [7, 8] introduced a convolution operator using generalized hypergeometric functions to study specific classes of univalent functions. Srivastava [9] also introduced a convolution operator with the Fox-Wright function to examine certain classes of univalent functions. Meanwhile, Baricz [10, 11], Orhan and Yagmur [12] as well as M.U.Din [1], explored the properties of Bessel, Struve, and Dini functions, respectively.

By exploring the Barnes-Mittage-Leffler functions' q-close-to-convex behaviors and their relationship to geometric constraints, this paper aims to highlight their importance and potential for future research, particularly in the context of fractional calculus and the geometry of special functions. Through continued investigation, these functions may lead to further advancements in both theoretical and applied mathematics.

Important Lemmas

Lemma 3.1[13]: Let $l \in A$ and $D_0 = 0$,$D_1 = 1$ and (a_m) be a sequence of natural numbers such that

$$D_m = [m]_q a_m = \frac{a_m(1-q)}{1-q}, \qquad \forall m \in N, q \in (0,1).$$

suppose,

$$1 \geq D_1 \geq D_2 \geq D_3 \geq, \ldots \geq D_m \geq, \ldots \geq 0.$$

Or

$$1 \leq D_1 \leq D_2 \leq D_3, \ldots \leq D_m \leq, \ldots \leq 2.$$

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in \kappa_{\hat{q},g} ,$$

Where,

$$g(z) = \frac{z}{1-z}.$$

Lemma 3.2[14]

Consider (a_m) be a series of actual numbers in which,

$$D_m = \frac{a_m(1-q^m)}{1-q}, \qquad \forall m \in N, q \in (0,1).$$

Let

$$1 \geq D_3 \geq D_5 \geq D_7, \ldots \geq D_{2m-1} \geq, \ldots \geq 0.$$

or

$$1 \leq D_3 \leq D_5 \leq D_7, \ldots \leq D_{2m-1} \leq, \ldots \leq 2.$$

Then

$$g(z) = z + \sum_{m=2}^{\infty} a_{2m-1} z^{2m-1} \in \kappa_{\hat{q},g} ,$$

Where

$$g(z) = \frac{z}{1-z^2}.$$

ISSN Online: 3007-3154
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Main Results

Geometric Properties of Barnes Mittage-Leffler Functions

The Barnes-Mittag-Leffler function is defined as:

$$E_{(k,v,a)}(s;z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(nk+v)(n+a)^s}, a,v,\in\square\setminus\overline{z}_o, s,z\in\square,\ \mathrm{Re}(k)>0.$$

$$E(s;z)=z+\sum_{n=1}^{\infty}\frac{\Gamma(v)a^s z^{n+1}}{\Gamma(nk+v)(n+a)^s}.$$

$$E(s;z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(v)a^s z^n}{\Gamma(k(n-1)+v)(n-1+a)^s}.$$

Theorem 4.1

Let $s,a,k,v\in\square^+$ and

$(1+q+q^2)\Gamma(k+v)(1+a)^s\leq(1+q)\Gamma(2k+v)(2+a)^s, q\in(0,1).$

Then, $E(s;z)$ is q-close-to-convex in the open unit d cisc with respect to star like function,

$$g(z)=\frac{z}{1-z}.$$

Proof

Consider

$$E(s;z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(v)a^s z^n}{\Gamma(k(n-1)+v)(n-1+a)^s}.$$

This expression can also be expressed as:

$$E(s;z)=z+\sum_{n=2}^{\infty}a_n z^n.$$

Where $a_{n=}\dfrac{\Gamma(v)a^s}{\Gamma(k(n-1)+v)(n-1+a)^s}.$

To prove that Barnes-Mittage-Leffler function is q-close-to-convex, we consider

$$B_n=\frac{1-q^n}{1-q}a_n, \forall n\in\square, q\in(0,1).$$

So that

$$B_n=\frac{1-q^n}{1-q}\frac{\Gamma(v)a^s}{\Gamma(k(n-1)+v)(n-1+a)^s}.$$

For $n=1$

$B_1=1.$

It can easily observe that $B_1=1$ and all values of B_n are positive for all positive integers.

Furthermore, from Lemma 3.1, we have

Put $n=2$,

$$B_2=\frac{1+q\Gamma(v)a^s}{\Gamma(k+v)(1+a)^s}<1.$$

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DIALOGUE SOCIAL SCIENCE REVIEW

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Next we will prove that

$$B_{n+1} < B_n, \left(n \in \mathbb{N} - \{1\}\right).$$

This implies that

$$\frac{1-q^{n+1}}{\Gamma(kn+v)(n+a)^s} \leq \frac{1-q^n}{\Gamma(k(n-1)+v)(n-1+a)^s}.$$

$$\left(1-q^{n+1}\right)\Gamma(k(n-1)+v)(n-1+a)^s \leq \left(1-q^n\right)\Gamma(kn+v)(n+a)^s.$$

For $n=3$, implies

$$\left(1+q+q^2\right)\Gamma(k+v)(1+a)^s \leq (1+q)\Gamma(2k+v)(2+a)^s.$$

Thus, if $\left(1-q^{n+1}\right)\Gamma(k(n-1)+v)(n-1+a)^s \leq \left(1-q^n\right)\Gamma(kn+v)(n+a)^s$, holds.

Then

$$B_{n+1} < B_n, \left(n \in \mathbb{N} - \{1\}\right).$$

Hence from Lemma 3.1, $E(s;z)$ is q- close-to-convex in the open unit disc with respect to star like function $g(z) = \frac{z}{1-z}$.

Theorem 4.2

Let $s,a,k,v \in \mathbb{R}^+$ and

$$(1+q+q^2+q^3)\Gamma(2k+v)(2+a)^s \leq \left(1+q+q^2\right)\Gamma(4k+v)(4+a)^s, q \in (0,1).$$

Then, $E(s;z)$ is q-close-to-convex in the open unit disc with respect to starlike function

$$g(z) = \frac{z}{1-z^2}.$$

Proof

Consider

$$E(s;z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(v)a^s z^n}{\Gamma(k(n-1)+v)(n-1+a)^s}.$$

This expression can also be expressed as:

$$E(s;z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Where $a_n = \frac{\Gamma(v)a^s}{\Gamma(k(n-1)+v)(n-1+a)^s}$.

To prove that Barnes-Mittage-Leffler function is q-close-to-convex, we consider

$$B_n = \frac{1-q^n}{1-q} a_n, \forall n \in \mathbb{N}, q \in (0,1).$$

So that

$$B_n = \frac{1-q^n}{1-q} \frac{\Gamma(v)a^s}{\Gamma(k(n-1)+v)(n-1+a)^s}. \quad (1)$$

For $n=1$

$$B_1 = 1.$$

It can easily observe that $B_1 = 1$ and all values of B_n are positive for all positive

integers.
Furthermore, from Lemma 3.2, we have
Put $n=3$ in (1)

$$B_3=\frac{\left(1+q+q^2\right)\Gamma(v)a^s}{\Gamma\left(2k+v\right)\left(2+a\right)^s}\leq 1.$$

Next we prove that

$$B_{2n+1}\leq B_{2n-1},\left(n\in\mathbb{N}-\{1\}\right).$$

Which implies that

$$\frac{\left(1-q^{2n+1}\right)}{\Gamma\left(2kn+v\right)\left(2n+a\right)^s}\leq\frac{1-q^{2n-1}}{\Gamma\left(k(2n-2)+v\right)\left(2n-2+a\right)^s}.$$

$$\Rightarrow\left(1-q^{2n+1}\right)\Gamma\left(k(2n-2)+v\right)\left(2n-2+a\right)^s\leq\left(1-q^{2n-1}\right)\Gamma\left(2kn+v\right)\left(2n+a\right)^s.$$

$$\Rightarrow\left(1-q^5\right)\Gamma\left(2k+v\right)\left(2+a\right)^s\leq\left(1-q^3\right)\Gamma\left(4k+v\right)\left(4+a\right)^s.$$

$$\Rightarrow\left(1+q+q^2+q^3\right)\Gamma\left(2k+v\right)\left(2+a\right)^s\leq\left(1+q+q^2\right)\Gamma\left(4k+v\right)\left(4+a\right)^s.$$

Thus if

$$\left(1-q^{2n+1}\right)\Gamma\left(k(2n-2)+v\right)\left(2n-2+a\right)^s\leq\left(1-q^{2n-1}\right)\Gamma\left(2kn+v\right)\left(2n+a\right)^s.$$

Then

$$B_{2n+1}\leq B_{2n-1},\left(n\in\mathbb{N}-\{1\}\right).$$

Hence from Lemma 3.2, $E(s;z)$ is q-close-to-convex in the open unit disc with respect to starlike function $g(z)=\frac{z}{1-z^2}$.

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