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Mapping Properties For Conic Regions Associated With Miller Ross Function

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Abstract
In this paper, our primary objective is to determine sufficient conditions under which the convolution operator $\mathbb{E}_{v,c}(z) = z\,E_{v,c}(z) * f(z)$ belongs to the classes $Uvc(k,\alpha)$ and $S_p(k,\alpha)$.

1. Introduction and preliminaries

Let A denote the class of functions of the form

(1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n ,$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. $S \subset A$ denote the subclass of functions that are univalent in U. The classes of star like and convex functions of order α are denoted by $S^*(\alpha)$ and $C(\alpha)$, respectively, and are defined as follows:

$$S^*(\alpha) = \{ f : f \in \dot{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in U, \alpha \in [0,1]$$

and

$$C(\alpha) = \{ f : f \in \dot{A} \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in U, \alpha \in [0,1].$$

Clearly

$$S^*(0) = S^* \text{ and } C(0) = C$$

These classes $S^*(\alpha)$ and $C(\alpha)$, were first introduced by by Robertson in 1936 (see [1, 2] for more details). The concept of Uniformly convex functions (UCV) and uniformly starlike functions (S_p) were later introduced by Goodman in 1991 [3, 4]. A function $f \in A$ is said to be uniformly convex if the image of every circular arc τ, contained within the open unit disc and centered at a point also inside the disc, is mapped onto a convex curve. This characterization was independently confirmed by Ma and Minda in 1992 [5] and by Rønning in 1993 [6] .

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Definition 1.1
A function $f \in A$ is uniformly convex in the open unit disc if and only if it satisfies the geometric condition that the image of every circular arc contained in the disc, with center also in the disc, is a convex curve. i.e.

(2) $$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|.$$

Definition 1.2
$f \in \dot{A}$ is in S_p if

(3) $$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|.$$

Next, we introduce the subclasses of k-uniformly convex functions of order α, along with a newly defined class associated with starlike functions. These function classes were introduced by Bharati et al. in 1997 [7].

Definition 1.3 A function $f \in \dot{A}$ is in $Uvc(k,\alpha)$ if and only if

(4) $$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > K\left|\frac{zf''(z)}{f'(z)}\right| + \alpha,\ \ z \in U,$$

where $0 \le k < \infty$ and $0 \le \alpha < 1$.
By applying the Alexander transform, we obtain the class $S_p(k,\alpha)$, defined as

Definition 1.4 A function $f \in Uvc(k,\alpha)$ if and only if $zf' \in S_p(k,\alpha)$
In 1993, Kenneth S. Miller and Bertram Ross introduced a special function known as the Miller-Ross function [10].

Definition 1.5 [8]: The Miller Ross function is expressed as
$\mathbb{E}_{v,c}(z) = z^v e^{cz} \Upsilon^*(v,cz)$,
Here $\Upsilon^*$is referred as incomplete gamma function.
$\mathbb{E}_{v,c}(z)$ is a solution of the ordinary differential equation
$Dy - cy = \frac{z^{v-1}}{\Gamma(v)}, v > 0.$
Applications for the Hadamard or Convolution product have multiple uses in the field of geometric function theory.

Definition 1.6 The Hadamard convolution of the functions of class $\dot{A}$ is defined by
$(f * g)z = z + \sum_{n=1}^{\infty} a_n b_n z^n$ (z∈ U)
here $f(z)$ and $g(z)$ are convergent power series in the open unit disc.
The **normalized form of Miller Ross function** is defined as follows:

(10) $$\mathbb{E}_{v,c}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(v+1)}{\Gamma(v+n)} z^n . c^{n-1}$$

Let f ∈ $\dot{A}$ given by (1)and g ∈ $\dot{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n ,$$

then Hadamard product (or convolution) of f and g is defined as:

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \qquad z \in U.$$

Now, we discuss about the convolution operator

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$$\mathbb{E}_{v,c}(z) = z\,E_{v,c}(z) * f(z)$$
$$= z + \sum_{n=2}^{\infty} A_n z^n$$

Where $A_n = \frac{\Gamma(v+1)}{\Gamma(v+n)} c^{n-1} . a_n$

In our present work we find some sufficient conditions under which the convolution operator $\mathbb{E}_{v,c}(z) = z\,\mathbb{E}_{v,c}(z) * f(z)$ belongs to the classes $Uvc(k,\alpha)$ and $S_p(k,\alpha)$.

Following lemmas will be helpful to prove our main results

Lemma 1.7 [7] A function $f \in \dot{A}$ is in $Uvc(k,\alpha)$ if it assure

(11) $$\sum_{n=2}^{\infty} n\{n(1+k)-(k+\alpha)\}|a_n| \le 1-\alpha.$$

Lemma 1.8 [7] A function $f \in \dot{A}$ is in $S_p(k,\alpha)$ if it assure

(12) $$\sum_{n=2}^{\infty} n\{n(1+k)-(k+\alpha)\}|a_n| \le 1-\alpha.$$

Lemma 1.9 [9] If $f \in P^{T(\eta)}{}_{\varphi}$ is defined in (3.7) , then

(13) $$|a_n| \le \frac{2|T|(1-\eta)}{1+\varphi(n-1)}$$

2. Main Results:

Theorem 2.1 Let $v > -1$, $c, z \in \mathbb{C}$ and $\alpha \in [0,1)$ with inequality such that

$$\frac{2(1-\eta)\cos\xi}{\varphi}\left\{(1+k)\frac{(2vc+2c-c^2)}{(v-c+1)^2}+(1-\alpha)\frac{c}{v+1-c}\right\} \le (1-\alpha)$$

If $f \in P_{\varphi}^{\tau(\eta)}$, $\varphi \in [0,1)$ and $\eta < 1$, then the convolution operator $\mathbb{E}_{v,c}(z) f(z) \in Uvc(k,\alpha)$.

Proof: Consider

$$\mathbb{E}(z) = zE(z) * f(z)$$
$$= z + \sum_{n=2}^{\infty} A_n z^n$$

Where $A_n = \frac{\Gamma(v+1)}{\Gamma(v+n)} c^{n-1} . a_n$

To show that the convolution $\mathbb{E}_{v,c}(z) f(z) \in Uvc(k,\alpha)$. From lemma 1.7, we prove that

$$\sum_{n=2}^{\infty} n\{n(1+k)-(k+\alpha)\}|A_n| \le 1-\alpha$$

Now

$$n\{n(1+k)-(k+\alpha)\}\frac{\Gamma(v+1)}{\Gamma(v+n)} c^{n-1} . a_n$$

$$\le 2(1-\eta)\cos\xi \sum_{n=2}^{\infty} n\{n(1+k)-(k+\alpha)\}\frac{\Gamma(v+1)c^{n-1}}{\Gamma(v+n)[1+\psi(n-1)]}$$

Since, $\frac{n}{1+\psi(n-1)} \le \frac{1}{\varphi}$, $\forall\, n \ge 2$, therefore we have

$$\sum_{n=2}^{\infty} n\{n(1+k)-(k+\alpha)\}\frac{\Gamma(v+1)}{\Gamma(v+n)} c^{n-1} . a_n$$

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$$\leq \frac{2(1-\eta)\cos\xi}{\varphi}\left\{\sum_{n=2}^{\infty} n(1+k)\frac{\Gamma(v+1)}{\Gamma(v+n)}c^{n-1} - \sum_{n=2}^{\infty}(k+\alpha)\frac{\Gamma(v+1)}{\Gamma(v+n)}c^{n-1}\right\}$$

$=$

$\frac{2(1-\eta)\cos\xi}{\varphi}\left\{(1+k)\sum_{n=2}^{\infty}(n-1+1)\frac{\Gamma(v+1)}{\Gamma(v+n)}c^{n-1} - (k+\alpha)\sum_{n=2}^{\infty}\frac{\Gamma(v+1)}{\Gamma(v+n)}c^{n-1}\right\}$

$= \frac{2(1-\eta)\cos\xi}{\varphi}\left\{(1+k)\sum_{n=2}^{\infty}(n-1)\frac{\Gamma(v+1)}{\Gamma(v+n)}c^{n-1} + (1-\alpha)\sum_{n=2}^{\infty}\frac{\Gamma(v+1)}{\Gamma(v+n)}c^{n-1}\right\}$

...(1)

By using inequality

$$\frac{\Gamma(v+1)}{\Gamma(v+n)} \leq \frac{1}{(v+1)^{n-1}}$$

So, (1) becomes

$$\sum_{n=2}^{\infty} n\{n(1+k)-(k+\alpha)\}\frac{\Gamma(v+1)}{\Gamma(v+n)}c^{n-1}.|a_n|$$

$$\leq \frac{2(1-\eta)\cos\xi}{\varphi}\left\{(1+k)\sum_{n=2}^{\infty}\frac{(n-1)c^{n-1}}{(v+1)^{n-1}} + (1-\alpha)\sum_{n=2}^{\infty}\frac{c^{n-1}}{(v+1)^{n-1}}\right\}$$

$$= \frac{2(1-\eta)\cos\xi}{\varphi}\left\{(1+k)\frac{c(v+1)}{(v-c+1)^2} + (1-\alpha)\frac{c}{v+1-c}\right\}$$

$\leq 1-\alpha.$

Hence proved.

Theorem 2.2 Let $v > -1$, $c > 0$ and $\alpha \in [0,1)$ with inequality

$$\frac{2(1-\eta)\cos\xi}{\varphi}\left\{\frac{(1+k)c}{v+1-c} - \frac{(k+\alpha)(e^c-c-1)}{c(v+1)}\right\} \leq 1-\alpha$$

If $f \in P_\varphi^{\tau(\eta)}$, $\varphi \in [0,1)$ and $\eta < 1$, then the convolution $\mathbb{E}_{v,c}f(z) \in S_p(k,\alpha)$.

Proof: Consider

$$\mathbb{E}_{v,c}\, f(z) = z\,\mathbb{E}_{v,c}(z) * f(z)$$

$$= z + \sum_{n=2}^{\infty} A_n z^n$$

Where $A_n = \frac{\Gamma(v+1)}{\Gamma(v+n)}c^{n-1}.a_n$

To show that convolution $\mathbb{E}_{v,c}f(z) \in S_p(k,\alpha)$, we use lemma 1.10, we get

$$\sum_{n=2}^{\infty}\{n(1+k)-(k+\alpha)\}A_n \leq 1-\alpha$$

Now

$$\sum_{n=2}^{\infty}\{n(1+k)-(k+\alpha)\}\frac{\Gamma(v+1)}{\Gamma(v+n)}c^{n-1}|a_n|$$

$$\leq \frac{2(1-\eta)\cos\xi}{1+\psi(n-1)}\sum_{n=2}^{\infty}\{n(1+k)-(k+\alpha)\}\frac{\Gamma(v+1)}{\Gamma(v+n)}c^{n-1}$$

Since, $\frac{n}{1+\psi(n-1)} \leq \frac{1}{\varphi}$, $\forall\, n \geq 2$, therefore we have

$\sum_{n=2}^{\infty}\{n(1+k)-(k+\alpha)\}A_n \leq$

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$$\frac{2(1-\eta)\cos\xi}{\varphi}\sum_{n=2}^{\infty}\{n(1+k)-(k+\alpha)\}\left\{\frac{1}{n}\frac{\Gamma(v+1)}{\Gamma(v+n)}c^{n-1}\right\}$$

$$=\frac{2(1-\eta)\cos\xi}{\varphi}\left\{(1+k)\sum_{n=2}^{\infty}\frac{\Gamma(v+1)}{\Gamma(v+n)}c^{n-1}-(k+\alpha)\sum_{n=2}^{\infty}\frac{\Gamma(v+1)}{n.\Gamma(v+n)}c^{n-1}\right\} \quad ....(1)$$

Using inequality

$$\frac{\Gamma(v+1)}{\Gamma(v+n)}\le\frac{1}{(v+1)^{n-1}}$$

And log-convexity property of Gamma function

$$\frac{\Gamma(v+1)\Gamma(k)}{\Gamma(v+k)}\le\frac{1}{v+1}$$

Therefore (1) becomes

$$\sum_{n=2}^{\infty}\{n(1+k)-(k+\alpha)\}|A_n| \le \frac{2(1-\eta)\cos\xi}{\varphi}\left\{(1+k)\sum_{n=2}^{\infty}\frac{c^{n-1}}{(v+1)^{n-1}}-(k+\alpha)\sum_{n=2}^{\infty}\frac{\Gamma(n)\Gamma(v+1)}{n.(n-1)!\Gamma(v+n)}c^{n-1}\right\}$$

$$\frac{2(1-\eta)\cos\xi}{\varphi}\left\{\frac{c}{v+1-c}(1+k)-(k+\alpha)\sum_{n=2}^{\infty}\frac{c^{n-1}}{(v+1)n!}\right\}$$

$$\frac{2(1-\eta)\cos\xi}{\varphi}\left\{(1+k)\frac{c}{v+1-c}-\frac{(k+\alpha)}{(v+1)}\frac{(e^{c}-c-1)}{c}\right\}$$

$$\le 1-\alpha,$$

By lemma 1.8. This completes the proof of theorem 2.2.

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